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ABSTRACT

Unit 10 in the SMSG's secondary school mathematics series is a student text covering the following topics in Algebra I: factors and exponents, radicals, polynomial and rational expressions, truth sets of open sentences, graphs of open sentences in two variables, systems of equations and inequalities, quadratic polynomials, and functions. (DT)

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STUDENT'S TEXT

UNIT NO. 10

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FIRST COURSE IN ALGEBRA

PART II

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SCHOOL MATHEMATICS STUDY GROUP

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School Mathematics Study Group

First Course in Algebra

Unit 10

First Course in Algebra

Student's Text, Part II

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Chapter 10

FACTORS AND EXPONENTS

10-1. Factors and Divisibility

Once upon a time, there was a farmer whose total property amounted to 11 cows. This farmer had three sons, and when he died, he left a will which provided that $\frac{1}{2}$ his cows should be left to Charles, $\frac{1}{4}$ of his cows to Richard, and $\frac{1}{6}$ of his cows to Oscar. The boys quarreled greatly over this, because none of them wanted the non-integral pieces of bovine matter which the will seemed to require. As they were arguing, along the road came a stranger, leading a cow which he was taking to market. The three boys confided their problem to him, and the stranger replied: "That's simple. Just let me give you my cow, and then try it." The boys were delighted, for they now had 12 cows instead of 11. Charles took half of these, 6; Richard took his quarter, that is 3; and Oscar his sixth, namely 2 cows. The 11 cows which the father had left were now happily divided; the stranger took his cow and went on his way.

So that you may not complain that the boys did not get exactly what the will provided, observe that each in fact got more, for $6 > \frac{11}{2}$, $3 > \frac{11}{4}$, and $2 > \frac{11}{6}$ (can you prove these inequalities?). However, there is something fishy about the problem, and it must be with the provisions of the will. What is it that made such an unusual solution possible?

This anecdote does have one mathematical conclusion at which we now want to look. For some reason, it was much easier to deal with 12 cows than with 11. And what was this reason? It was that 6 and 4 and 2 all divide into 12 exactly, while none of these seems to divide into 11 exactly. And this is an important distinction between 12 and 11: there are many numbers which divide exactly into 12, but very few that divide into 11.

It is a bit clumsy to write "divide into exactly" all the time, and so we shall use a more compact mathematical term for this. We shall say that 6 is a "factor" of 12 because $6 \times 2 = 12$; similarly, 4 is a factor of 12 (because $4 \times 3 = 12$), and so on. Is 3 also a factor of 12? Is 2?

The number 5, however, is not a factor of 12, because we cannot find another integer such that 5 times that integer equals 12. Of course 1 and 12 are also factors of 12. Given any positive integer, 1 and the integer itself are factors of that integer; because such factors are always present, they are not very interesting. So we shall call 2 and 3 and 4 and 6 proper factors of 12; these and 1 and 12 are all factors. The number 11, however, does not have any proper factors, because no positive integer other than 1 and 11 is a factor of 11. Now we are ready for a more precise definition of a factor, remembering that a factor of n is one of two integers whose product is n.

> The integer m is a factor of the integer n if $mq = n$, where q is an integer. If the integer q does not equal 1 or n, we say that m is a proper factor of n.

Does it follow from this definition that if m is a proper factor of n, then m also cannot equal 1 or n ?

Since 3 is a factor of 18, then is $\frac{18}{3}$ a factor of 18? Is it true that if m is a factor of n, then $\frac{n}{m}$ is a factor of n? Is the same true for proper factors? How can you tell?

Since 5 is a factor of 15, we say that 5 divides 15. In general if m and n are positive integers and if m is a factor of n, we say that m divides n, or n is divisible by m. We shall say that 0 is divisible by every integer, but 0 does not divide any number.

Problem Set 10-1a

For each of the questions below, if the answer is "Yes", write the number in factored form as in the definition. If the answer is "No", justify in a similar way.

Example: Is 5 a factor of 45? Yes, since $5 \times 9 = 45$.
Is 5 a factor of 46? No, since there is no integer q such that $5q = 46$.

1. Is 2 a factor of 24?
2. Is 3 a factor of 24?
3. Is 5 a factor of 24?
4. Is 6 a factor of 24?
5. Is 9 a factor of 24?
6. Is 13 a factor of 24?
7. Is 12 a factor of 24?
8. Is 24 a factor of 24?
9. Is 13 a factor of 91?
10. Is 30 a factor of 510?
11. Is 12 a factor of 204?
12. Is 10 a factor of 100,000?
13. Is 3 a factor of 10,101?
14. Is 6 a factor of 20,202?
15. Is 12 a factor of 40,404?

If any of the following numbers are factorable (i.e. have proper factors), find such a factor, and find the product which equals the given number and uses this factor.

Example: $69 = 3 \times 23$
67 is not factorable

16. 85
17. 51
18. 52
19. 29
20. 93
21. 92
22. 37
23. 94
24. 55
25. 61
26. 23
27. 123
28. 57
29. 65
30. 122
31. 68
32. 95
33. 129
34. 141
35. 101

Let us now consider how you can tell whether 2 is a factor of a given number. Which of the numbers with which you just worked did have 2 as a factor? Is there an easy way to tell whether or not 2 is a factor of a number?

Let us now look at the numbers 5 and 10. When is 5 a factor of some integer? You have probably known for some time that the decimal notation for every multiple of 5 ends in either a 5 or a 0, and every integer whose decimal notation ends in a 5 or a 0 is a multiple of 5. Also the decimal representation of every multiple of 10 ends in 0, and every decimal numeral which ends in 0 is a multiple of 10. We can now look at this in a slightly different way: a number has 10 as a factor if and only if it has both 2 and 5 as factors. Numbers which have 5 as a factor must have decimals which end in 5 or 0, and numbers which have 2 as a factor must be even; hence, if a number is to have *both* 2 and 5 as a factor its decimal form must end in 0. Can you formulate what we have just said in terms of two sets and members common to both?

Problem Set 10-1b

Think about a test to check whether a number is divisible by 4, and also a test for divisibility by 3. The following examples should give you some real hints on the solutions - but don't be disappointed if a simple rule for 3 to be a factor of a number escapes you for a moment, for it is rather tricky.

1. Divisibility by 4: Which of the following numbers have 4 as a factor? 28, 128, 228, 528, 3028; 6, 106, 306, 806, 2006; 18, 118, 5618; 72, 572? Do you see the test? How many digits of the number do you have to consider?

2. Divisibility by 3: Which of the following numbers have 3 as a factor? 27, 207, 2007, 72, 702, 270; 16, 106, 601, 61, 1006. How about 36, (observe that $3 + 6 = 9$), 306, 351, 315, 513, 5129, (observe that $5 + 1 + 2 + 9 = 17$), 32122? We write

$$\begin{aligned} 2358 &= 2(1000) + 3(100) + 5(10) + 8(1) \\ &= 2(999 + 1) + 3(99 + 1) + 5(9 + 1) + 8(1) \\ &= 2(999) + 3(99) + 5(9) + 2(1) + 3(1) + 5(1) + 8(1) \\ &= \big(2(111) + 3(11) + 5(1)\big)9 + (2 + 3 + 5 + 8) \\ &= (222 + 33 + 5)9 + (2 + 3 + 5 + 8). \end{aligned}$$

The expression $(222 + 33 + 5)9$ is divisible by 3 (Why?); is $2 + 3 + 5 + 8$ divisible by 3? Notice that the sum of the digits is the key to divisibility by 3. Try to formulate this as a rule.

3. If a number is divisible by 9, is it divisible by 3? If a number is divisible by 3, is it divisible by 9?

4. If you know a test for both 2 and 3, what would be a test for 6?

5. Answer the following questions and in each case tell which divisibility tests made your work easier.
   (a) Is 3 a factor of 101,001?
   (b) Is 3 a factor of 37,199?
   (c) Is 6 a factor of 151,821?
   (d) Is 15 a factor of 91,215?
   (e) Is 12 a factor of 187,326,648?

10-2. Prime Numbers

We have been talking about factors of positive integers over the positive integers, in the sense that when we write

$$mq = n$$

we accept only positive integers for m, n and q. We could, of course, accept negative integers, or any rational numbers, or even any real numbers, as factors. But if you consider these possibilities for a moment, you will see that they do not add much to our understanding. If, for example, you permit negative integers as factors, do you really find anything new? For example, -2, 2, -3, 3 are factors of 6. How are the factors which involve negative integers related to those which involve positive integers only?

You get a different picture if you accept all rational numbers as possible factors of positive integers. The rational number $\frac{2}{7}$, for example, would be a factor of 13, in this extended sense, because $(\frac{2}{7})(\frac{91}{2}) = 13$. Can you think of any non-zero rational number, in fact, which would not be a factor of 13 in this sense? Try $-\frac{17}{3}$, for example. Since $(-\frac{17}{3})(-\frac{39}{17}) = 13$, we find that $-\frac{17}{3}$ is also a rational factor of 13. Is the situation any different if you permit factoring over the real numbers?

You see that if you try factoring positive integers over the rational numbers or over the real numbers, then every number other than zero becomes a factor of every number. Such a kind of factoring, therefore, would not add much to our understanding of the structure of the real number system, and so we shall not consider it further. Usually factoring over the positive integers gives us the most interesting results, and so when we speak of "factoring" a positive integer, we shall always mean over the positive integers.

Problem Set 10-2a

1. We have listed below a set of positive integers less than or equal to 100. Cross out the numbers for which 2 is a proper factor and write a 2 below each of these numbers. (For example, $\cdots$, $\cancel{8}_2$, 9, $\cancel{10}_2$, $\cdots$)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What is the first number after 2 which has not been crossed out? It should be 3. Now cross out all numbers which have 3 as a proper factor and write a 3 below each of the numbers. If a number has already been crossed out with a 2 do not cross it out again but skip it. What now is the first number after 3 which has not been crossed out? It should be 5. Cross out numbers which have as proper factor the number 5. Continue the process. After the fifth step your picture should look like this.

1	2	3	$\cancel{4}_2$	5	$\cancel{6}_2$	7	$\cancel{8}_2$	$\cancel{9}_3$	$\cancel{10}_2$
11	$\cancel{12}_2$	13	$\cancel{14}_2$	$\cancel{15}_3$	$\cancel{16}_2$	17	$\cancel{18}_2$	19	$\cancel{20}_2$
$\cancel{21}_3$	$\cancel{22}_2$	23	$\cancel{24}_2$	$\cancel{25}_5$	$\cancel{26}_2$	$\cancel{27}_3$	$\cancel{28}_2$	29	$\cancel{30}_2$
31	$\cancel{32}_2$	$\cancel{33}_3$	$\cancel{34}_2$	$\cancel{35}_5$	$\cancel{36}_2$	37	$\cancel{38}_2$	$\cancel{39}_3$	$\cancel{40}_2$
41	$\cancel{42}_2$	43	$\cancel{44}_2$	$\cancel{45}_3$	$\cancel{46}_2$	47	$\cancel{48}_2$	$\cancel{49}_7$	$\cancel{50}_2$
$\cancel{51}_3$	$\cancel{52}_2$	53	$\cancel{54}_2$	$\cancel{55}_5$	$\cancel{56}_2$	$\cancel{57}_3$	$\cancel{58}_2$	59	$\cancel{60}_2$
61	$\cancel{62}_2$	$\cancel{63}_3$	$\cancel{64}_2$	$\cancel{65}_5$	$\cancel{66}_2$	67	$\cancel{68}_2$	$\cancel{69}_3$	$\cancel{70}_2$
71	$\cancel{72}_2$	73	$\cancel{74}_2$	$\cancel{75}_3$	$\cancel{76}_2$	$\cancel{77}_7$	$\cancel{78}_2$	79	$\cancel{80}_2$
$\cancel{81}_3$	$\cancel{82}_2$	83	$\cancel{84}_2$	$\cancel{85}_5$	$\cancel{86}_2$	$\cancel{87}_3$	$\cancel{88}_2$	89	$\cancel{90}_2$
$\cancel{91}_7$	$\cancel{92}_2$	$\cancel{93}_3$	$\cancel{94}_2$	$\cancel{95}_5$	$\cancel{96}_2$	97	$\cancel{98}_2$	$\cancel{99}_3$	$\cancel{100}_2$

Did the picture change from the fourth step to the fifth step? Why or why not? If you are having difficulty with this question perhaps it would help if you would consider the first number crossed out in each step. How far would the set of numbers have to be extended before the picture after the fifth step would be different from the picture after the fourth step?

In the set of the first 100 positive integers, you have crossed out all the numbers which have proper factors. Thus the remaining numbers have no proper factors. We call each of these numbers, other than 1, a prime number.

> A prime number is a positive integer greater than 1 which has no proper factors.

Is it possible to find all the prime numbers in the set of positive integers by the method we have just used (called the Sieve of Eratosthenes)? Is it possible to find all the prime numbers less than some given positive integer by this method? What is the next prime number after 97?

Problem Set 10-2b

1. What is the largest prime number less than 100? less than 200? less than 300?
2. What is the largest prime proper factor of numbers less than 100? 200? 300?
3. Note that no prime number greater than 7 was needed in crossing out non-prime numbers less than 100 in the Sieve of Eratosthenes. What is the largest prime number needed in crossing out all non-prime numbers less than 200? less than 300?

10-3. Prime Factorization

Let us now return to the Sieve of Eratosthenes and see what else we can learn from it. Consider, for example, the number 63. It is crossed out, and hence 63 is not a prime. When did we cross out 63? We see from the diagram that 63 was crossed out when we were working with 3. This means, if you stop and think about it, that 3 is the smallest prime factor of 63. (Actually, it follows from what we have just said that 3 is the smallest proper factor of 63, regardless of the "prime". Do you see why?)

Since 3 is the smallest prime factor of 63, let us divide it out. We obtain 21, and once again we can look in our chart to see if 21 is a prime. We find that it is not, and that in fact 3 is a factor of 21. Divide 21 by 3, and you obtain 7; if you look for 7 on the chart you find that it is not crossed out, so that 7 is a prime and can be factored no further. What is it that we have learned from all this? We have obtained 63 as 3 times 3 times 7; and the significance of this is that these factors of 63 are all primes. In other words, we have succeeded in writing 63 as a product of prime factors: $63 = 3 \times 3 \times 7$.

Let us try the same procedure again with 60. What prime were you considering when you crossed out 60? If you divide 60 by this prime, what do you obtain? Continue the process. What factorization of 60 as a product of primes do you obtain?

Problem Set 10-3a

1. Using the Sieve of Eratosthenes, write each of the following numbers as a product of prime factors:

   84, 16, 37, 48, 50, 18, 96, 99, 78, 47, 12.

2. What positive integers have the following prime factorizations, respectively:

   (a) $2 \times 2 \times 7$, (b) $2 \times 3 \times 5$, (c) 7×11,
   (d) $2 \times 3 \times 3 \times 3$, (e) 7×7, (f) $2 \times 2 \times 3 \times 3$?

A positive integer, you see, can be thought of as "made up" of a number of prime factors. Thus 63 is made up of two 3's and one 7; 60 is made up of two 2's , one 3, and one 5. We shall have many uses for this "prime factorization", as it is called, of a positive integer. But now we face a problem: How do we do the same thing for a number which is not on our diagram? If you are asked for the prime factorization of 144, you might perhaps consider extending the diagram from 100 to 144. But suppose you are asked for 1764?

Maybe you can figure out a way to do the same thing without a picture of the sieve before you. What, after all, went on when you constructed the sieve? You first marked all numbers which were multiples of 2 with a "2"; the first number not marked was 3, and you proceeded to mark all numbers which were proper multiples of 3, and so on. Then came 5, and then 7. What were these numbers 2, 3, 5, 7, etc.? They were just the prime numbers. Thus, if 2 was a proper factor, it was crossed out when we were working with multiples of 2; if 2 was not a factor, but 3 was, then "3" was crossed out when we were working with multiples of 3, and so forth. If the number had no proper factors, i.e. was prime, it was not crossed out at all.

Let us now do the same thing without the sieve, say with 1764. We first try 2. (It is convenient to start with the smallest prime factor.) Is 2 a factor of 1764? By the tests which we learned, the answer is, "Yes"; 1764 = 2 × 882. So now let us try 882, as if we had the sieve before us. Is 2 a factor of 882? Yes; and 882 = 2 × 441. Now let us work on 441. Is 2 a factor of 441? No, it isn't; so if our sieve had gone as far as 441, this number would not have been crossed out when we considered multiples of 2. The next prime after 2 is 3, and so we must test next whether or not 441 is a multiple of 3. If you check 441 for divisibility by 3, you find that 3 divides (4 + 4 + 1), and hence 3 is a factor of 441, so that it would have been crossed out in the sieve when we tested multiples of 3. We now obtain 441 as 3 × 147. There is no sense trying the factor 2 on 147, since if 2 were a factor of 147, it would also have been a factor of 441 (why?). But 3 divides 147, and we obtain $147 = 3 \times 49$. 49, in turn, is 7×7, and 7 is a prime number, so that the job is finished. To summarize: We have found that $1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$, and this is the prime factorization which we are looking for.

All this writing is rather clumsy; a more compact way to write it is

$$\begin{array}{r|l} 1764 & 2 \\ 882 & 2 \\ 441 & 3 \\ 147 & 3 \\ 49 & 7 \\ 7 & 7 \\ 1 & \end{array}$$

Here the smallest prime factor at any stage is to the right of the line, and the quotient of the dividend by the smallest prime factor is underneath the dividend. The prime factorization can be read off from the factors to the right of the line.

Problem Set 10-3b

1. What is the smallest prime factor of 115, of 135, of 321, of 484, of 539, of 143?
2. Find the prime factorization of each of the following numbers: 98, 432, 258, 625, 180, 1024, 378, 729, 825, 576, 1098, 486, 3375, 3740, 1311, 5922, 1008, 5005, 444, 5159, 1455, 2324.

You may have noticed that we have been speaking of "the" prime factorization of a positive integer, as if we were sure that there was only one such factorization. Do you suppose that this is actually true? Can you give any convincing reasons why this should be the case? The fact that every positive integer has exactly one prime factorization is often called the Fundamental Theorem of Arithmetic.

10-4. Adding and Subtracting Fractions

One of the many uses of prime factorization of integers is in addition and subtraction of fractions. It is easy to add or subtract two fractions if their denominators are the same. We have already found it possible to use the property of 1 to change one fraction to an equal fraction having a different denominator. In this way we can change fractions to be added or subtracted into fractions having the same denominator.

To make addition of fractions as easy as possible it is desirable that this common denominator shall be the least common multiple of the denominators. We define the least common multiple of two or more given integers as the smallest integer which is divisible by all the given integers.

Consider the problem of simplifying

$$\frac{1}{4} - \frac{3}{10} - \frac{4}{45} + \frac{1}{6}$$

We can readily see that one common multiple of the denominators is their product $4 \times 10 \times 45 \times 6$, or 10,800. This seems very large. Perhaps what we have learned about prime factorization can help us to find the smallest integer which has 4 and 10 and 45 and 6 as factors.

Consider the prime factors of each denominator:

$$\begin{aligned} 4 &= 2 \times 2, \\ 10 &= 2 \times 5, \\ 45 &= 3 \times 3 \times 5, \\ 6 &= 2 \times 3. \end{aligned}$$

Since 4 must be a factor of the common denominator, this denominator must, in its own prime factorization, contain at least two 2's. In order that 10 be a factor of the denominator, the latter's prime factorization must contain a 2 and a 5; we already have a 2 by the previous requirement that 4 be a factor, but we must also include a 5 now. To summarize what we have so far: in order that both 4 and 10 be factors of the denominator, the prime factorization of the denominator must contain at least two 2's and one 5.

Next we must have 45 as a factor. This means we have to supply two factors of 3 as well as the two 2's and the 5 we already have; we don't need to supply another 5 because we already have one. Finally, to accommodate the factor 6, we need both a 2 and a 3 in the factorization, but we already have each of these.

Conclusion: The denominator will have the prime factorization $2 \times 2 \times 3 \times 3 \times 5$. We need each of these factors, and any more would be unnecessary. What unnecessary factors did the denominator 10,800 contain?

Now that we have found the least common denominator, we can complete the problem of changing each of the fractions in our problem so that it has this denominator. What will $\frac{1}{4}$ become? An easy way to do this is to use the factored form of the least common denominator and the factored form of 4. 4 contains two 2's and nothing more, while the common denominator contains two 2's, two 3's and one 5. Thus, to change 4 into the desired denominator, we have to multiply by two 3's and one 5 to supply the missing factors.

$$\frac{1}{4} = \frac{1}{2 \times 2} = \frac{1}{2 \times 2} \times \frac{3 \times 3 \times 5}{3 \times 3 \times 5} = \frac{45}{2 \times 2 \times 3 \times 3 \times 5} .$$

Similarly,

$$\frac{3}{10} = \frac{3}{2 \times 5} = \frac{3}{2 \times 5} \times \frac{2 \times 3 \times 3}{2 \times 3 \times 3} = \frac{54}{2 \times 2 \times 3 \times 3 \times 5} .$$

Can you now do the same with $\frac{4}{45}$ and $\frac{1}{6}$? If you have completed the arithmetic correctly, you will have

$$\frac{1}{4} - \frac{3}{10} - \frac{4}{45} + \frac{1}{6} = \frac{45 - 54 - 16 + 30}{2 \times 2 \times 3 \times 3 \times 5}$$

$$= \frac{5}{2 \times 2 \times 3 \times 3 \times 5} = \frac{1}{2 \times 2 \times 3 \times 3} = \frac{1}{36} .$$

What are the advantages of this way of doing the problem? One advantage is the avoidance of big numbers; the denominator is left in factored form until the very end, and you see that we never had to handle any number larger than 54. Another advantage of having the denominator in factored form is that we need only test the resulting numerator for divisibility by the factors of the denominator in order to change the fraction to "lowest terms".

Problem Set 10-4

1. Find the following sums.
   - (a) $\frac{2}{9} + \frac{1}{15}$
   - (b) $\frac{3}{14} - \frac{4}{35}$
   - (c) $-\frac{1}{12} + \frac{4}{26}$
   - (d) $-\frac{5}{12} - \frac{7}{18}$
   - (e) $\frac{1}{85} + \frac{3}{51}$
   - (f) $-\frac{20}{57} - \frac{7}{95}$
   - (g) $\frac{5}{21} - \frac{3}{91}$
   - (h) $\frac{3x}{8} + \frac{5x}{36}$
   - (i) $\frac{1}{6} + \frac{3}{20} - \frac{2}{45}$
   - (j) $\frac{3k}{10} + \frac{2k}{28} - \frac{k}{56}$
   - (k) $\frac{3a}{5} + \frac{7a}{75} - \frac{5a}{63}$
   - (ℓ) $\frac{x}{3} + \frac{5x}{8} - \frac{11}{70} + \frac{3}{20}$

2. Are the following sentences true?
   - (a) $\frac{8}{15} < \frac{13}{24}$
   - (b) $\frac{3}{16} < \frac{11}{64}$
   - (c) $\frac{14}{63} < \frac{6}{27}$

3. In each of the following pairs, which number is greater?
   - (a) $\frac{1}{7}$ or $\frac{1}{2} - \frac{1}{3}$
   - (b) $\frac{4}{15}$ or $\frac{7}{27}$
   - (c) $\frac{5}{12}$ or $\frac{5}{13}$

4. You have learned in Chapter 5 that for any pair of numbers a and b, exactly one of the following must hold: $a > b$, $a = b$, or $a < b$. Put in the symbol for the correct relation for the following pairs of numbers.

(a) $\frac{6}{27}$, $\frac{5}{28}$ (c) $\frac{6}{16}$, $\frac{9}{24}$

(b) $\frac{2}{3}$, $\frac{5}{7}$ (d) $(\frac{1}{2} + \frac{1}{3}), (\frac{11}{12} - \frac{1}{13})$

5. John and his brother Bob each received an allowance of $1.00 per week. One week their father said, "I will pay each of you $1.00 as usual or I will pay you in cents any number less than 100 plus its largest prime proper factor. If you are not too lazy, you have much to gain." What number should they choose?

6. Suppose John's and Bob's father forgot to say proper factor. How much could the boys gain by their father's carelessness?

*7. A man is hired to sell suits at the AB Clothing Store. He is given the choice of being paid $200 plus $\frac{1}{12}$ of his sales or a straight $\frac{1}{3}$ of sales. If he thinks he can sell $600 worth of suits per month, which is the better choice? Suppose he could sell $700 worth of suits, which is the better choice? $1000? What should his sales be so that the offers are equal?

10-5. Some Facts About Factors

Suppose that you were looking for two integers with the property that their sum is 22 and their product is 72. One way to find them would be to try all possible ways of factoring 72, and keep looking until you found a pair that met the condition. We are now going to see, however, that factors of numbers have properties which allow us to rule out many possibilities without actually trying them. The prime factorization of 72 is $2 \times 2 \times 2 \times 3 \times 3$. The two factors of 72 which we are seeking must use up the three 2's and two 3's in the prime factorization of 72. Suppose three 2's were all in one factor, and no 2's in the other; that is, either $(2 \times 2 \times 2)(3 \times 3)$ or $(2 \times 2 \times 2 \times 3)(3)$ or $(2 \times 2 \times 2 \times 3 \times 3)(1)$, then one factor would be even, while the other factor would be odd, because it contained no 2's. But the sum of an even and an odd number is odd, and 22 is not odd, that is,

$$(2 \times 2 \times 2) + (3 \times 3) = 17 \quad \text{or}$$
$$(2 \times 2 \times 2 \times 3) + 3 = 27 \quad \text{or}$$
$$(2 \times 2 \times 2 \times 3 \times 3) + 1 = 73$$

So this split of 72 won't work; we will have to split the three 2's between the two factors, and thus put two 2's in one factor, and one 2 in the other.

Next, let us look at the 3's. Do we split the two 3's, or do they both go into one of the two factors? We know 22 does not have 3 as a factor; but if we were to split the two 3's in 72 between the two factors of 72, then each would have 3 as a factor, and then the sum would have 3 as a factor. The sum could certainly not be 22.

We have thus learned that the two factors of 72 must be "put together" as follows: one factor contains two 2's while the other factor contains one 2; one factor contains both 3's, while the other contains no 3's. There are only two possibilities; the two 3's go either with the one 2 or with the two 2's, that is, either $(2 \times 2 \times 3 \times 3)(2)$ or $(2 \times 3 \times 3)(2 \times 2)$. But two 2's with two 3's makes 36, which is clearly too big, so that the two 3's go with the one 2 (making 18) and the other two 2's (making 4) form the other factor. Since $(2 \times 3 \times 3) + (2 \times 2) = 22$ and $(2 \times 3 \times 3)(2 \times 2) = 72$, we have our answer.

The kind of reasoning which we have just done depends on two ideas, namely: if 2 is a factor of one of two numbers, and 2 is a factor of their sum, then 2 is a factor of the other number; and if 3 is a factor of one of two numbers and 3 is not a factor of their sum, then 3 is not a factor of the other number.

Let us first prove a similar theorem.

Theorem 10-5a. For positive integers b and c, if 2 is a factor of both b and c, then 2 is a factor of (b + c).

Proof: $2q = b$, q an integer, because 2 is a factor of b;
$2p = c$, p an integer, because 2 is a factor of c.

Therefore,

$$2q + 2p = b + c, \qquad \text{(Why?)}$$
$$2(q + p) = b + c, \qquad \text{distributive property}$$

Since $q + p$ is an integer,
2 is a factor of $(b + c)$.

For example, Theorem 10-5a guarantees that since 2 is a factor of both 6 and 16, it follows that 2 is a factor of $(6 + 16)$. Since 7 is a factor of both 21 and 35, do you think it follows that 7 is a factor of $(21 + 35)$? If we replace 2 in Theorem 10-5a by any positive integer a, we can prove the general result.

Theorem 10-5b. For positive integers a, b and c, if a is a factor of both b and c, then a is a factor of $(b + c)$.

The proof of this theorem will be an exercise in the next set of problems.

Another useful theorem is

Theorem 10-5c. For positive integers a, b and c, if a is a factor of b, and a is not a factor of $(b + c)$, then a is not a factor of c.

Proof: Assume a is a factor of c; then a is a factor of both b and c and, hence, is a factor of $(b + c)$. (Why?) But this contradicts the given fact that a is not a factor of $(b + c)$. Hence, a is not a factor of c.

Since 3 is a factor of 15, and 3 is not a factor of $(15 + 8)$, we are certain that 3 is not a factor of 8.

A third theorem useful in dealing with factors is

Theorem 10-5d. For positive integers a, b and c, if a is a factor of b, and a is a factor of $(b + c)$, then a is a factor of c.

The proof is left to the student in an exercise.

Problem Set 10-5

1. The prime factorization of 12 is $2 \times 2 \times 3$. What two numbers, whose product is 12, have an even sum? odd sum? Can you find another odd sum? (Remember that 1 is a factor of every positive integer.) Can 3 be a factor of any possible sum?

2. The prime factorization of 36 is $2 \times 2 \times 3 \times 3$. Find two numbers whose product is 36 and whose sum will be divisible by 3 but not 2; divisible by 2 but not 3; divisible by neither 2 nor 3.

3. The prime factorization of 150 is $2 \times 3 \times 5 \times 5$. Find two numbers whose product is 150 and (a) whose sum is even; (b) whose sum is divisible by 5; (c) whose sum is not divisible by 5.

4. Write the prime factorization of 18. Find two numbers whose product is 18 and whose sum is 9; 11.

5. Write the prime factorization of the first number in each of the following and use it to find two numbers whose product and whose sum are as indicated below.
   (a) Product is 288 and sum is 34
   (b) " " 972 " " " 247
   (c) " " 216 " " " 217
   (d) " " 330 " " " 37
   (e) " " 500 " " " 62
   (f) " " 270 " " " 39

6. The perimeter of a rectangular field is 68 feet and the area is 225 square feet. If the length and width are integers, find them.

7. Prove Theorem 10-5b.

8. Prove Theorem 10-5d.

9. Show that if y is a positive integer, then y is a factor of $(3y + y^2)$. (Hint: Use Theorem 10-5b)

10. For what positive integer x is 3 a factor of $6 + 4x$? (Hint: apply Theorem 10-5d. How many values of x can you find?)

11. If 3 boys shovel snow from sidewalks and charge 50¢ for a store and $1.50 for a house, how many walks of stores and how many walks of houses must they shovel in order to split the money evenly? *What if there were 4 boys?

12. (a) 3 is a factor of 39 and 39 is a factor of 195. Does it follow that 3 is a factor of 195?

    (b) 3 is a factor of 39 and 5 is a factor of 20. Does it follow that $3 + 5$ is a factor of $39 + 20$?

    (c) 3 is a factor of 39 and 5 is a factor of 20. Does it follow that $3 \cdot 5$ is a factor of $39 \cdot 20$?

    (d) 3 is a factor of 39 and 3 is a factor of 27. Does it follow that 3^2 is a factor of $39 \cdot 27$?

    (e) 3 is a factor of 39 and 3 is a factor of 27. Does it follow that 3 is a factor of $39 + 27$?

    (f) 3 is a factor of 39. Does it follow that 3^2 is a factor of 39^2?

    (g) 3^2 is a factor of 15^2. Does it follow that 3 is a factor of 15?

    (h) 3 is a factor of 39 and 5 is a factor of 135. Does it follow that 3 is a factor of 135?

13. Prove the theorems:

    (a) For positive integers a, b, c, if a is a factor of b, and b is a factor of c, then a is a factor of c.

(b) For positive integers a, b, c, d, if a is a factor of b, and c is a factor of d, then ac is a factor of bd.

(c) For positive integers a, b, c, if a is a factor of b, and a is a factor of c, then a^2 is a factor of bc.

(d) For positive integers a and b, if a is a factor of b, then a^2 is a factor of b^2.

14. Which theorem in Problem 13 justifies the following:

(a) 25 is a factor of $(35)(15)$.

(b) 6 is a factor of $(30)(7)$.

(c) $(13)^2$ is a factor of $(39)(26)$.

(d) 49 is a factor of $(14)(35)$.

(e) c^2 is a factor of $(5c)(9c)$ if c is a positive integer.

(f) $20ab$ is a factor of $(15)(24)ab$ if ab is a positive integer.

10-6. Introduction to Exponents.

We have seen that we can write a positive integer factored into its prime factors, so that, for example,

$$288 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 .$$

This notation is somewhat inconvenient because it is so lengthy; it would not be necessary to write the "2" five times if there were some way, more compact than $2 \times 2 \times 2 \times 2 \times 2$, of writing "five 2's multiplied together."

As a first step in this direction, you already know that 3×3 can be written as 3^2. This is pronounced "3 squared"; the "3" indicates that we are going to multiply 3's together, and the "2" that we are going to multiply two of them. How would we write $2 \times 2 \times 2 \times 2 \times 2$ similarly? The number 288 can thus be written in factored form more compactly as $2^5 \times 3^2$.

In an expression of the form a^n, we need some way of describing the numbers involved. The "a", which indicates which number we are going to use as a factor several times, is called the base; the "n", which indicates how many of the factors "a" we are going to use, is called the exponent. Thus a^n means a number consisting of n equal factors a; a^n is called a power, or more precisely, the nth power of a. We can write

$$a^n = \underbrace{a \times a \times \ldots \times a.}_{n \text{ factors}}$$

a^2 is read "a squared" or "a square".

a^3 is read "a cubed" or "a cube".

a^n is read "a to the nth power", or just "a to the nth".

Problem Set 10-6a

1. Can you guess how the words "squared" and "cubed" originated? If a side of a square is 5 inches long, how large, in square inches, is its area?
2. Find the prime factorization of each of the following numbers, using exponents wherever appropriate. 64, 60, 80, 48, 128, 81, 49, 41, 32, 15, 27, 29, 56, 96, 243, 432, 512, 576, 625, 768, 686.
3. In describing the number a^n, what kind of number must n be? Must a be?

*4. The expression a^b can be thought of as defining a binary operation which, for any two positive integers a and b produces the number a^b. What does it mean to ask if this operation is commutative? Is it? What would it mean to ask whether or not this operation is associative?

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Let us extend our notions about exponents. Since we know that the set of real numbers is closed under multiplication, it must be true that $a^3 \cdot a^2$ names a real number. Is there a simpler name? Since a^3 means that a is a factor three times and a^2 means that a is a factor twice, it follows that $a^3 \cdot a^2$ has a as a factor five times. That is,

$$a^3 \cdot a^2 = \overbrace{\underbrace{a \cdot a \cdot a}_{\text{3 factors}} \cdot \underbrace{a \cdot a}_{\text{2 factors.}}}^{\text{5 factors}} = a^5$$

Write simpler names for each of the following: $a^2 \cdot a^3$; $b^3 \cdot b^3$; $3^3 \cdot 3^4$; $(x^2)(x^5)$; $a^4 \cdot a^3 \cdot a^2$; $c^5 \cdot c^8$; $a^2 \cdot b^3$; $2^2 \cdot 3^3$. Suppose we consider the number $a^m \cdot a^n$, where m and n are positive integers.

$$a^m\, a^n = \overbrace{\underbrace{a \times a \times a \times \dots \times a}_{m \text{ factors}} \times \underbrace{a \times a \times a \times \dots \times a}_{n \text{ factors}}}^{m + n \text{ factors}} = a^{m+n}$$

Does it seem reasonable, therefore, to say that $a^m\, a^n$ and a^{m+n} are names for the same number?

Have you noticed that we have been talking about a^2, a^3, a^5, a^7, etc., that is, forms of the type a^n, where n is a positive integer; but we have not mentioned a^1? Certainly, 1 is a positive integer and we shall define $a^1 = a$.

Problem Set 10-6b

1. Write simpler names for the following.

   Example: $(9x^2)(3x^4) = (3^2x^2)(3x^4)$

   $= 3^3x^6$.

   (a) $m^3\, m^{11}$

   (b) $(x^3)(x^9)$

(c) $(2x^2)(x)$

(d) $(2x)(2x^3)$

(e) $(2x)(2^3x^3)$

(f) $(27a)(3^4a^3)$

(Hint: replace 27 by its prime factorization.)

(g) $(16a^2)(32a^8)$

(h) $(x^{2a})(x^a)$

(i) $3^4 \cdot 3^2$

(j) $3^4 \cdot 2^3$

(k) $2^5 \cdot 3^2 \cdot 5 \cdot 2^2 \cdot 3^3 \cdot 5^2$

(ℓ) $2^a \cdot 2^{4a}$

(m) $(3a^2b^3)(3^2ab^2)$

(n) $(3k^2t)(3m^2t)$

In Problems 2 - 12 tell which sentences are true and which are false and show why in each case.

2. $2^3 + 3^3 = 5^3$
3. $2^3 \cdot 3^3 = 6^3$
4. $2^3 + 3^3 = 6^3$
5. $2^3 \cdot 3^3 = 6^6$
6. $2^3 \cdot 3^3 = 6^9$
7. $2^3 + 2^3 = 2^6$
8. $2^3 + 2^3 = 2^4$
9. $3^3 + 3^3 = 3^4$
10. $3^3 + 3^3 + 3^3 = 3^4$
11. $4^3 + 4^3 + 4^3 = 4^4$
12. $4^3 + 4^3 + 4^3 + 4^3 = 4^4$

13. Write other names for:

(a) $2^3(2^2 + 2)$

(b) $x^2(2x^3 + x^2)$

(c) $2x^3(2x^2 - 4x^3)$

(d) $-3a^4(3^2a^3 - 3^3a^2)$

(e) $(a^2 + 2a^3)(a - a^2)$

10-7. Further Properties of Exponents.

Now let us examine the fraction $\frac{a^5}{a^3}$, $a \neq 0$. Is there a simpler name for this fraction? From the meaning of a^5 and a^3 it is evident that

$$\begin{aligned} \frac{a^5}{a^3} &= \frac{a \times a \times a \times a \times a}{a \times a \times a} \\ &= a \times a \times \frac{a \times a \times a}{a \times a \times a} \\ &= a^2 \end{aligned}$$

Write simpler names for: $\frac{x^5}{x^2}$; $\frac{b^2}{b^3}$; $\frac{c^6}{c}$; $\frac{3^7}{3^2}$; $\frac{a^2}{a^2}$; $\frac{m^3}{m^2}$; where none of the variables has the value 0. Can you generalize the results? Suppose we consider $\frac{a^5}{a^3}$ again, but reason in this way:

$$a^5 = a^3 \cdot a^2 \text{ , because } a^m \cdot a^n = a^{m+n} .$$

Then

$$\begin{aligned} \frac{a^5}{a^3} &= \frac{1}{a^3} \cdot (a^3 \cdot a^2) \\ &= \left(\frac{1}{a^3} \cdot a^3\right) a^2 && \text{(Why?)} \\ &= 1 \cdot a^2 && \text{(Why?)} \\ &= a^2 && \text{(Why?)} \end{aligned}$$

That is, if $m > n$,

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{1}{a^n} (a^n \cdot a^{m-n}) \\ &= \left(\frac{1}{a^n} \cdot a^n\right) a^{m-n} \\ &= 1 \cdot a^{m-n} \\ &= a^{m-n} \end{aligned}$$

We specify that $m > n$ because we want $m - n$ to be a positive integer.

If $m = n$,

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{a^m}{a^m} \\ &= 1 \end{aligned}$$

If $m < n$,

$$\begin{aligned}
\frac{a^m}{a^n} &= a^m\left(\frac{1}{a^m \cdot a^{n-m}}\right) \\
&= a^m\left(\frac{1}{a^m} \cdot \frac{1}{a^{n-m}}\right) \\
&= \left(a^m \cdot \frac{1}{a^m}\right) \frac{1}{a^{n-m}} \\
&= 1 \cdot \frac{1}{a^{n-m}} \\
&= \frac{1}{a^{n-m}}
\end{aligned}$$

To summarize: When $(a \neq 0)$

If $m > n$ then $\frac{a^m}{a^n} = a^{m-n}$. For example $\frac{6^5}{6^3} = 6^2$.

If $m = n$ then $\frac{a^m}{a^n} = 1$. For example $\frac{6^5}{6^5} = 1$.

If $m < n$ then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$. For example $\frac{6^5}{6^9} = \frac{1}{6^4}$

Problem Set 10-7a

1. Simplify:

(a) $\frac{2^3}{2^2}$ (c) $\frac{2^3}{2^3}$ (e) $\frac{2^{16}}{2^{12}}$ (g) $\frac{2^2 \cdot 3^4}{2^5 \cdot 3^4}$

(b) $\frac{2^2}{2^3}$ (d) $\frac{2^3}{3^3}$ (f) $\frac{2^{10}}{2^{13}}$ (h) $\frac{2^3 \cdot 3^5}{2^2 \cdot 3^7}$

In Problems 2 - 7 simplify each expression. (We assume that no variable will have the value 0.)

2. (a) $\frac{a^3}{a}$ (b) $\frac{m}{m^4}$ (c) $\frac{z^3}{z^3}$ (d) $\frac{b^3}{b^{11}}$

3. (a) $\frac{2x^6}{2^3x^2}$ (b) $\frac{3^2b^6}{3b^4}$ (c) $\frac{5b^4}{5b^4}$ (d) $\frac{4^2a}{4a^2}$

4. (a) $\frac{a^2b^3c}{a^4b^3c^4}$ (b) $(a^2b^3c)(a^4b^3c^4)$ (c) $a^2b^3c + a^4b^3c^4$.

5. (a) $\frac{(5x)(5x)}{5^3x^3}$ (b) $\frac{5x(5 + x)}{5^3x^3}$ (c) $\frac{(5x)(5x)}{5x}$

6. (a) $\frac{36a^2b^3}{8a^5b}$ (b) $\frac{36a^2b^3}{6a^2b^2}$ (c) $\frac{36a^2b^3}{7ab^9}$

7. (a) $\frac{288x^2y^3}{48x^6y^6}$ (b) $\frac{54x^2y^3}{153x^6a^6}$ (c) $\frac{63x^2y^3}{28a^6b^6}$

In Problems 8 - 12 tell which sentences are true and which are false and show why.

8. $\frac{3^2}{2^2} = \frac{3}{2}$

9. $\frac{6^3}{3^3} = 2$

10. $\frac{3^4}{2^4} = (\frac{3}{2})^4$

11. $\left(\frac{4^3}{3^3}\right)(\frac{3}{4})^3 = 1$

12. $\frac{6^3}{3^3} = 2^3$

13. Why must we be careful to avoid 0 as the value of the variables in Problems 2 - 7?

Having three properties of exponents for handling division is never as satisfactory as just one which will do the same job. It happens that it is possible to reduce all three to just one, namely:

$$\frac{a^m}{a^n} = a^{m - n} ,$$

if we drop the condition $m > n$. Let us work some problems in two ways, first, using whichever property of the last section is appropriate and second, using

$$\frac{a^m}{a^n} = a^{m-n} \quad .$$

It is convenient to tabulate the results.

Complete the table.

Compare $\frac{a^7}{a^3} = a^{7-3} = a^4$ with $\frac{a^7}{a^3} = a^{7-3} = a^4$

Compare $\frac{a^3}{a^3} = 1$ with $\frac{a^3}{a^3} = a^{3-3} = a^?$

Compare $\frac{a^3}{a^5} = \frac{1}{a^{5-3}} = \frac{1}{a^2}$ with $\frac{a^3}{a^5} = a^{3-5} = a^?$

Compare $\frac{a^4}{a^4} = 1$ with $\frac{a^4}{a^4} = a^{4-4} = a^?$

Compare $\frac{a^2}{a^3} = \frac{1}{a}$ with $\frac{a^2}{a^3} = a^{2-3} = a^?$.

We have extended notions of numbers in many instances before; can you now extend your notion of exponents? Examine the above table carefully to answer the following questions:

$$a^0 = \ ?$$

$$a^{-1} = \ ?$$

$$a^{-2} = \ ?$$

Do zero and negative exponents make any sense in our definition of $a^n = a\ a\ a \cdots$ to n factors? Of course, it is senseless to think of a as a factor (-3) times. But the above comparisons suggest a way to write just one property of exponents for division. If we define, for m and n positive integers and $a \neq 0$,

$$a^0 = 1,$$

and

$$a^{-n} = \frac{1}{a^n}, \ a \neq 0,$$

then

$$\frac{a^m}{a^n} = a^m \cdot \frac{1}{a^n}$$

$$= a^m \cdot a^{-n}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad ,$$

Example 1

$$\frac{7^3}{7^5} = 7^{3-5}$$

$$= 7^{-2}$$

$$= \frac{1}{7^2}$$

Is this the same result you get using the former definition?

Now that we have a meaning for a negative exponent and for a zero exponent, the properties

$$a^m a^n = a^{m+n} \quad \text{and} \quad \frac{a^m}{a^n} = a^{m-n}$$

hold for any integers m and n, whether positive, zero, or negative.

Example 2

$$\frac{x^{-2}y^{-3}}{x^4\ y^{-2}} = \left(\frac{x^{-2}}{x^4}\right)\left(\frac{y^{-3}}{y^{-2}}\right) \quad , \quad x \neq 0,\ y \neq 0,$$

$$= x^{-2-4}\ y^{-3-(-2)}$$

$$= x^{-6} \cdot y^{-1}$$

$$= \frac{1}{x^6} \cdot \frac{1}{y}$$

$$= \frac{1}{x^6 y}$$

Example 3

$$\begin{aligned}\frac{10^3 \times 10^{-4}}{10^{-5}} &= \frac{10^{3+(-4)}}{10^{-5}}\\ &= \frac{10^{-1}}{10^{-5}}\\ &= 10^{-1-(-5)}\\ &= 10^4\end{aligned}$$

Problem Set 10-7b

1. Simplify each of the following, first by the single property

$$\frac{a^m}{a^n} = a^{m-n}$$

and also in a form using only positive exponents. (Assume none of the variables takes on the value 0.)

Example: $\frac{a^7}{a^9} = a^{-2} = \frac{1}{a^2}$

(a) $\frac{3^5}{3^3}$

(b) $\frac{3^5}{3^8}$

(c) $\frac{3^8}{3^8}$

(d) $\frac{b^4}{b^2}$

(e) $\frac{b^7}{b^{10}}$

(f) $\frac{10^5}{10^6}$

(g) $\frac{10^5 \times 10^2}{10^8}$

(h) $\frac{10^2 \times 10^4}{10^3}$

(i) $\frac{10^4 \times 10^3}{10^2 \times 10^5}$

(j) $\frac{a^4b^3}{a^7b}$

(k) $\frac{36x^2y^4}{8x^5y}$

(ℓ) $\frac{3m^4}{m^9}$

(m) $\frac{t^3}{3t^5}$

2. Simplify to a form with only positive exponents. (Assume none of the variables takes on the value 0.)

(a) $\dfrac{10^4 \times 10^{-2}}{10^2}$

(b) $\dfrac{10^3 \times 10^2}{10^{-2}}$

(c) $.007 \times 10^4 \times 10^{-4}$

(d) $\dfrac{12a^4b}{3a^7b^2}$

(e) $\dfrac{2x^2y^{-2}}{4x^2y^2}$

(f) $\dfrac{2x^2y^{-2}}{4^2x^2y^2}$

(g) $\dfrac{3^2 \times 2^{-3}}{2^3 \times 3^{-2}}$

(h) $\dfrac{10^3 \times 10^{-4} \times 10^0}{10^2 \times 10^{-3}}$

(i) $\dfrac{2^{-3}x^{-2}y^4}{2^{-2}x^2y^{-1}}$

3. The distance from the earth to the sun in miles is approximately 93,000,000.

(a) How many millions of miles is this?

(b) How many "ten millions" of miles?

(c) Is 9.3×10^7 another name for 93,000,000?

4. Simplify to a form with only positive exponents.

(a) $10^2(10^4 + 10^{-2})$

(b) $3a^2(3a + a^{-1})$

(c) $3a^2(3^{-1}a + 3a^{-2})$

(d) $(a + a^{-1})(a + a^{-1})$

(e) $(a + a^{-1})(a - a^{-1})$

5. In the following, what value of n makes the sentence true?

(a) $10^3 \times 10^3 = 10^n$

(b) $10^{-1} \times 10^{-1} = 10^n$

(c) $10^{-4} \times 10^{-4} = 10^n$

(d) $10^7 \times 10^7 = 10^n$

(e) $10^n \times 10^n = 10^8$

(f) $10^n \times 10^n = 10^{-6}$

(g) $10^n \times 10^n = 10^{18}$

(h) $10^n \times 10^n = 10^{-4}$

6. If n is a positive integer and $a \neq 0$, prove that

$$a^n = \frac{1}{a^{-n}}.$$

*7. If m and n are any integers, then show that

$$a^m a^n = a^{m+n}$$

includes all cases of multiplication and division of powers. (Hint: The case of $\frac{a^p}{a^q}$ can be considered as $a^{p+(-q)}$.)

What is the meaning of $(ab)^3$? We know ab names a number, and we also know that a number cubed means that the number is a factor three times. Therefore, $(ab)^3$ must mean $(ab)(ab)(ab)$. By the commutative and associative properties of multiplication for real numbers we know that

$$(ab)(ab)(ab) = (aaa)(bbb) = a^3b^3$$

Thus

$$(ab)^3 = a^3b^3$$

Write another name for $(\frac{a}{b})^3$, using similar reasoning. Write another name for $(a^2b^3)^3$ using similar reasoning. In general, we have

$$(ab)^n = a^nb^n.$$

Problem Set 10-7c

In Problems 1 - 8 simplify (assuming no variable has the value 0) and write answers with positive exponents only.

1. (a) $(3a^3)^2$ (b) $3(a^3)^2$ (c) $(3a^2)^3$ (d) $3a^{(3^2)}$

2. (a) $\frac{5x^2}{15xy^2}$ (b) $\frac{(5x)^2}{15xy^2}$ (c) $\frac{5x^2}{15(xy)^2}$

3. (a) $\frac{(-3)^2a}{9}$ (b) $\frac{-3^2a}{9}$ (c) $\frac{(-3a)^2}{9}$ (d) $\frac{(-3a)^3}{9}$

4. (a) $\frac{(2y^2)^3(2y)}{(2y)^3(2y^2)}$ (b) $\frac{(2y^2)^5}{(2y^2)^5}$ (c) $\frac{(2y)^5}{2y^5}$

5. (a) $\frac{-7^2z^{15}}{49z^{30}}$ (b) $\frac{(-7)^2z^{15}}{49z^{30}}$ (c) $\frac{-7^2z^{30}}{-49z^{15}}$

6. (a) $\left(\frac{28a^3}{45a}\right)\left(\frac{3}{4}\right)^2$ (b) $\left(\frac{63a^2}{243a^5}\right)\left(\frac{54a^7}{14a^4}\right)$ (c) $\left(\frac{37(-a)^3}{3a}\right)\left(\frac{2a^{11}}{7(-a)^4}\right)$

7. (a) $\frac{x^{2a}}{x^a}$ (b) $x^{2a}\cdot x^a$ (c) $(x^{2a})^3$

8. (a) $\dfrac{\frac{90(ab)^2}{16a^3}}{\frac{81ab^3}{108b}}$ (b) $\dfrac{\frac{(xy)^2}{xy^2}}{\frac{x^2y}{x^2y^2}}$

9. Is each of the following true? Give reasons for each answer.

(a) $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$

(b) $\frac{2}{3} = \frac{2^2}{3^2}$

(c) $\left(\frac{5a}{7b}\right)^2 = \frac{5^2a^2}{7^2b^2}$

(d) $\frac{5a^2}{7b^2} = \frac{5^2a^2}{7^2b^2}$

(e) 3^3 is a factor of $(3^3 + 3^5)$.

(f) 3^2 is a factor of $(6^2 + 9^2)$.

(g) $(2x + 4y^2)$ is an even number, if x and y are positive integers.

10. (1) Take a number, (2) double it, (3) then square the resulting number. Now start again: (1) take the original number, (2) square it, (3) then double the resulting number.

(a) Is the final result the same in both processes?

(b) Using a variable, show whether or not the two procedures lead to the same result.

11. (a) Consider a square whose side is s units long. What is its area? Now consider a square with sides twice the length of the sides of the original square. Write and simplify a phrase (in terms of s) for the area of the larger square. The area of the larger square is how many times that of the smaller?

    (b) In a similar manner compare the area of the original square with the area of a square with side three times as long.

    (c) If you have not already done so, show by sketching the squares involved that the results of parts (a) and (b) seem reasonable.

12. Simplify the following, that is, change to a form involving one indicated division.

Example:

$$\frac{5}{3x^2} + \frac{11}{6xy} - \frac{4}{9y^2} = \frac{5}{3x^2}\cdot\frac{3\cdot 2y^2}{3\cdot 2y^2} + \frac{11}{6xy}\cdot\frac{3xy}{3xy} - \frac{4}{9y^2}\cdot\frac{2x^2}{2x^2}$$

$$= \frac{30y^2 + 33xy - 8x^2}{3^2\cdot 2x^2y^2}$$

$$= \frac{30y^2 + 33xy - 8x^2}{18x^2y^2}$$

(Notice that $3^2\cdot 2\cdot x^2y^2$ is the least common multiple of $3x^2$, $6xy$, and $9y^2$ because $3^2\cdot 2\cdot x^2y^2$ is the smallest set of factors which contains $3x^2$, $3\cdot 2xy$, and $3\cdot 3y^2$.)

(a) $\frac{5}{6a} + \frac{9}{8a}$

(b) $\frac{5}{2x^2} - \frac{2}{3x^2} + \frac{1}{6x}$

(c) $\frac{x}{6a^3} - \frac{y}{5a^2} - \frac{z}{2a}$

(d) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

(e) $\frac{11}{35a^2} + \frac{13}{25ab} - \frac{7}{5b^2}$

13. Prove: If a^2 is odd and if a is an integer, then a is odd. (Hint: Assume a is even and obtain a contradiction.)

14. Prove: If a^2 is even and if a is an integer, then a is even. (Hint: Assume a is odd.)

15. Let a be 2, b be -2, c be 3, d be -3. Then determine the value of:

(a) $-2a^2b^2c^2$

(b) $(-2abc)^2$

(c) $\frac{-4a^4d}{6b^2a^3}$

(d) $\frac{-6a^{12}b^{16}c^{20}}{2a^{10}b^{18}c^{22}}$

(e) $\frac{a^3 + b^3}{a^3b^3}$

(f) $\frac{(a + b + c)^2}{a^2 + b^2 + c^2}$

16. Multiply:

Example: $(a^2 - 3)(a^2 - 2a + 1) = a^2(a^2 - 2a + 1) - 3(a^2 - 2a + 1)$

$= a^4 - 2a^3 + a^2 - 3a^2 + 6a - 3$

$= a^4 - 2a^3 - 2a^2 + 6a - 3$

(a) $(x^2 + 1)(x^3 + x^2 + 1)$

(b) $(2a^3 - b^2)(3a^2 - 2b^2)$

(c) $(2x - 3y)^2$

(d) $(a + b)^3$

Review Problems

1. Find the following sums:

(a) $\frac{1}{13} + \frac{3}{26}$

(b) $\frac{1}{4} - \frac{1}{3} + \frac{1}{6}$

(c) $\frac{27}{35} - \frac{19}{21}$

(d) $\frac{5m}{12} - \frac{7m}{18} - \frac{1}{2}$

(e) $\frac{a}{44} + \frac{2a}{33} + \frac{3a}{22}$

(f) $\frac{2}{3a^2} - \frac{5}{12a} + \frac{1}{4}$

2. Simplify to "lowest terms".

   (a) $\frac{102}{2^2 \cdot 3 \cdot 5}$

   (b) $\frac{2 \cdot 5^2 \cdot 7 \cdot a^2}{450a}$

   (c) $\frac{51\ x^3y}{85\ xy}$

   (d) $\frac{142857}{999999}$

3. What is the next prime number after 129?

4. Four times an integer is ten more than twice its successor. What is the integer?

5. If the domain of the variable is the set of prime numbers, find the truth set of the following:

   (a) $\frac{1}{3}(2x - 99) = 29$

   (b) $\frac{x}{3} + \frac{5}{12} = 12 + \frac{1}{4}x$

   (c) $3x^2 < 123$

   (d) $|x - 10| < 3$

6. Simplify:

   (a) $(2a)(3a^2)$

   (b) $\frac{2xy}{8x^2}$

   (c) $m^2 \cdot m^{-2}$

   (d) $\frac{2^2 \cdot 2^3}{2^4}$

   (e) $\frac{3^5a^4}{3^2a}$

   (f) $\frac{10^3 \times 10^{-1}}{10^{-2}}$

7. Write the indicated products as indicated sums.

   (a) $a^2(a + 1)$

   (b) $xy^2(x^2 + y^3)$

   (c) $(2x + 1)\ 3x^2$

   (d) $(-mn)(m - n)$

   (e) $(a^2 + b^2)(a + b)$

   (f) $x^{-1}(x^2 + x^3)$

   (g) $(a + b)(a^{-1} + b^{-1})$

   (h) $(x^2y + 1)(xy^2 + 2y)$

8. If n is a positive integer, which of the following are even numbers, which are odd, and which may be either?

(a) n^2

(b) n^3

(c) $2n$

(d) $2n + 1$

(e) $(2n)^2$

(f) $(2n + 1)^2$

(g) $4n^2$

(h) $2n - 1$

(i) $2^{10} + 3^{10}$

(j) $2^{10} + 6^{10}$

9. Which of the following are non-negative for any real number n ?

(a) n^2

(b) $(-n)^3$

(c) $(-n)(-n)$

(d) $-n^3$

(e) $(-n)^2$

(f) $(n^2)(n^2)$

(g) n^4

(h) $(-n)^4$

(i) $-n^4$

(j) $-|n^2|$

(k) $!-n^3|$

10. Two squares differ in area by 27 square units. A side of the larger is one unit longer than a side of the smaller. Write and solve an equation to find the length of the side of the smaller square.

11. For 27 days Bill has been saving nickels and dimes for summer camp expenses. He finds he has 41 coins the value of which is \$3.35. If he has more dimes than nickels, how many nickels does he have?

12. Sam has five hours at his disposal. How far can he ride his bicycle into the surrounding hills at the rate of 8 miles per hour and return at the rate of 12 miles per hour?

Chapter 11

RADICALS

11-1. Roots

After we studied the operation of addition, it was possible to define its inverse operation, subtraction. What operation did we define as the inverse of multiplication? In Chapter 10 we considered the operation of squaring a number. This operation also has an inverse.

Let us review for a moment the process of finding the square of a number.

If $x = 17$, then what is the value of x^2?

If $x = .3$, then what is the value of x^2?

If $x = -2wa^2$, then what is x^2?

Now let us consider the same kind of question in the opposite direction.

What is the truth set of $x^2 = 49$?

of $x^2 = .09$?

of $x^2 = -4$?

In the second group of questions we are finding, for example, a number whose square is 49. This is the inverse operation to squaring, and is called finding a square root. One number whose square is 49 is certainly 7, and hence 7 is a square root of 49. Since it is also true that $(-7)^2 = 49$, it follows that -7 is also a square root of 49. Our notation and our terminology have to be chosen so that they will keep these two square roots distinct; people usually call the positive square root of a number b "the square root of b", and denote it by $\sqrt{b}$.

Let us now summarize this discussion.

> If b is a positive real number, and $a^2 = b$, then a is a square root of b. If a is a square root of b, so is $-a$. The positive square root of b is denoted by $\sqrt{b}$ and is commonly called "the" square root of b. The negative square root of b is then $-\sqrt{b}$.

We also define $\sqrt{0} = 0$, in which case there is only one square root.

Problem Set 11-1a

1. Simplify:

   (a) $\sqrt{4}$

   (b) $-\sqrt{121}$

   (c) $\sqrt{(-3)^2}$ (careful)

   (d) $\sqrt{2.25}$

   (e) $\sqrt{\frac{49}{9}}$

   (f) $\sqrt{.81}$

   (g) $\sqrt{4} + \sqrt{9} - \sqrt{25} + 3$

   (h) $2\sqrt{\frac{49}{4}} - 3\sqrt{\frac{64}{9}}$

2. Is the sentence "$\sqrt{x^2} = x$" true if

   (a) x is 3? (b) x is -3?

3. Is the sentence "$\sqrt{x^2} = |x|$" true if

   (a) x is 3? (b) x is -3?

4. If a and b are positive real numbers and $a < b$, prove that $\sqrt{a} < \sqrt{b}$. Hint: Exactly one is true: $\sqrt{a} = \sqrt{b}$, $\sqrt{a} > \sqrt{b}$, $\sqrt{a} < \sqrt{b}$. (Why?) Show that the first two lead to contradictions.

5. Is it possible that $\sqrt{x^2} + 2 = 1$ for some value of x? Explain.

6. Determine the square root of $(2x-1)^2$ if

   (a) $x < \frac{1}{2}$, (b) $x > \frac{1}{2}$, (c) $x = \frac{1}{2}$.

*7. Consider the following "proof" that all numbers are equal. If a and b are any real numbers, then

$$|a-b| = |b-a|,$$

$$(a-b)^2 = (b-a)^2,$$

$$a - b = b - a,$$

$$2a = 2b,$$

$$a = b.$$

Which step of this "proof" is faulty, and why?

If $x = 2$, then what is the value of x^3?

If $x = -.3$, then what is the value of x^3?

If $x = \frac{1}{2}a^2$, then what is x^4?

Again, in the other direction, we can ask:

What is the truth set of $x^3 = 8$?

of $x^3 = -0.027$?

of $x^4 = 16$?

Following the same procedure as before, we can say that a is a cube root of b if $a^3 = b$. We write $a = \sqrt[3]{b}$. Notice from the above examples that while we were not able to take square roots of negative numbers, since both negative and positive numbers have positive squares, it is perfectly possible to take cube roots of negative numbers, since the cube of a negative number is negative.

On the other hand, we seem to be able to find only one number whose cube is 8, namely 2, while we have seen that numbers other than 0 have two square roots when they have any at all. Within the framework of the real numbers, this is indeed correct; in the coming years, you will find that by extending the kinds of numbers we are willing to use, negative numbers will have square roots too, and every non-zero number will have three cube roots.

Problem Set 11-1b

In Problems 1 - 8 simplify:

1. (a) $\sqrt[3]{27}$ (b) $\sqrt[4]{81}$ (c) $\sqrt[5]{243}$
2. (a) $\sqrt[3]{8}$ (b) $\sqrt[3]{1000}$ (c) $\sqrt[3]{729}$
3. (a) $\sqrt[3]{x^3}$ (b) $\sqrt{x^2}$ (c) $\sqrt[4]{x^4}$
4. (a) $\sqrt[3]{-125}$ (b) $\sqrt[3]{-8y^3}$ (c) $\sqrt[3]{(-8)(-y)^3}$
5. (a) $\sqrt[3]{(-13)^3}$ (b) $\sqrt{(-13)^2}$ (c) $\sqrt[3]{(-13)^6}$
6. (a) $\sqrt[3]{.008}$ (b) $\sqrt[3]{\frac{27}{1000}}$ (c) $\sqrt[3]{.216}$
7. (a) $\sqrt[3]{-\frac{125}{8}}$ (b) $\sqrt[3]{\frac{64}{b^3}}$ (c) $\sqrt{\frac{a^2}{b^6}}$
8. (a) $\sqrt[3]{64c^6}$ (b) $\sqrt[3]{(x-3y)^3}$ (c) $\sqrt{\sqrt{81}}$
9. What is the relation between $\sqrt[4]{16}$ and $\sqrt{4}$? Between $\sqrt[4]{10,000}$ and $\sqrt{100}$? Can you guess a relation between fourth roots and square roots that seems to be true?

*10. Write a definition for fourth roots. For nth roots, where n is a positive integer. For what values of n do you think negative numbers will have real nth roots? How do you suppose the property of positive numbers of having two real square roots and one real cube root extends to nth roots?

11-2. Radicals

The symbol $\sqrt{}$ is called the radical sign; an expression which consists of a phrase and a radical sign over it is called a radical. Thus $\sqrt{3x^2}$ is a radical.

Let us now return to square roots. Thus far, we have not attempted to take the square root of n unless we were able, with more or less difficulty, to recognize n as the square of some simple number or expression. (We call n a perfect square in this case.) Let us now consider the case of a square root which we cannot recognize immediately, such as, for example, $\sqrt{2}$. What kind of question do we want to ask about this? For instance, if we were given the expression $\sqrt{\frac{4}{9}} + 1$, we would not leave it in this form because it involves an indicated operation which can be performed. It can be simplified to read $\frac{2}{3} + 1$, or just $\frac{5}{3}$. What if the expression were $\sqrt{2} + 1$?

Let us recall what happened in the case of $\sqrt{\frac{4}{9}} + 1$. We discovered that $\sqrt{\frac{4}{9}}$ was a rational number which we could combine with the rational number 1 and obtain the simpler expression $\frac{5}{3}$. Can we do something similar with $\sqrt{2} + 1$? We certainly do not know of any rational number whose square is 2, but we have not yet proved that there is no such number. The time has come to settle this question once and for all.

Theorem 11-2. $\sqrt{2}$ is irrational.

Discussion:

Before we begin the proof of this theorem, let us think about the problem involved. We want to prove that $\sqrt{2}$ is irrational, that is, that a number whose square is 2 cannot be rational. How does one prove that something does not have a certain property? For example, we proved in Chapter 9 that 0 has no reciprocal. How did we do this? We assumed that 0 does have a reciprocal, and showed that this assumption leads to a contradiction. If an assumption leads us to a contradiction, the assumption must be false; and if it is false that 0 has a reciprocal, then it has no reciprocal. Let us try the same reasoning here.

Proof:

Suppose that there is a rational number, say $\frac{a}{b}$, where a and b are positive integers, such that $\left(\frac{a}{b}\right)^2 = 2$. We can certainly insist that a and b have no common factor, for if they did, we could remove such a factor from the fraction $\frac{a}{b}$.

If
$$\left(\frac{a}{b}\right)^2 = 2,$$
then
$$\frac{a^2}{b^2} = 2, \qquad \text{(Why?)}$$
and
$$a^2 = 2b^2. \qquad \text{(Why?)}$$

This says that a^2 is an even number. (Why?) Then a itself is an even number. (You proved this fact in a problem in Chapter 10, page 280.)

If a is even then $a = 2c$, where c is another integer. (Why?) If we replace a by $2c$ in our last equation, we obtain
$$4c^2 = 2b^2,$$
$$2c^2 = b^2.$$

By the same argument which we just gave for a, we know that b must now be even, since its square is even. Hence, we have shown that both a and b must have been even. But a and b were chosen to have no common factor, and this certainly does not permit a and b to have the common factor 2. Thus, we have a contradiction; that is, the assumption that $\sqrt{2}$ is rational has led us to a contradiction, and the assumption must have been false. Hence, $\sqrt{2}$ is irrational. This completes the proof.

Notice, incidentally, an interesting difference between a proof by contradiction, such as we have just done, and other types of proof which you have seen during the course. In the direct proof, there is a specific fact which you are trying to establish, and you proceed to work with whatever facts you are given and with the properties of the real numbers until the fact you are seeking is before you. You concentrate on creating the statement you desire from statements which you have assumed to be true. In a proof by contradiction, on the other hand, you add to your list of things with which you work the denial of what you want to prove, and then keep deriving results until a contradiction appears. You don't know ahead of time just where this contradiction is coming from, but you keep working until you find one. This contradiction proves that you made a mistake in denying what you wanted to show, and thus what you wanted to show must have been true all along.

It is possible to establish in a similar way that the square root of any positive integer which is not a perfect square is irrational. Among the integers from 1 through 10, for example, this tells us that only 1, 4, and 9 have rational square roots, while the others have irrational square roots. Try to show that, for example, $\sqrt{3}$ is irrational.

Problem Set 11-2

1. Since all integers are rational numbers, the fact that $\sqrt{2}$ is not an integer is actually included in Theorem 11-2. Try to show directly that $\sqrt{2}$ is not an integer.
2. You learned in Chapter 1 that between any two points on the number line, there are infinitely many points labelled with rational numbers. Do you think that between any two points on the positive half of the number line there are infinitely many points whose coordinates are not only rational, but also perfect squares? Between the rational numbers $\frac{1}{2}$ and $\frac{1}{3}$ find two rational numbers which are perfect squares.
3. Prove that $\frac{1}{2}\sqrt{2}$ is irrational. (Hint: Assume it is rational and arrive at a contradiction.)
4. Prove that $\sqrt{2} + 3$ is irrational.

*5. Prove that $\sqrt{5}$ is irrational. (Hint: follow the same reasoning as in the proof that $\sqrt{2}$ is irrational. You may use the fact that if 5 divides p^2 and p is an integer, then 5 divides p .)

11-3. Simplification of Radicals

We observe that there can be only one positive number a which is a square root of n. For if there were a second such positive number, b, which is not the same as a, then either $a < b$ or $b < a$. (Why?) In these two cases, respectively, we would have $a^2 < b^2$ or $b^2 < a^2$. Thus, the squares of a and b could not both equal n.

Let us consider next the product of two square roots, say $\sqrt{2}$ and $\sqrt{3}$. Does this product equal some simpler expression? We can draw a conclusion about $\sqrt{2} \cdot \sqrt{3}$ by examining $(\sqrt{2} \cdot \sqrt{3})^2$.

$(\sqrt{2} \cdot \sqrt{3})^2 = (\sqrt{2})^2(\sqrt{3})^2 = 2 \cdot 3 = 6$.

Thus $\sqrt{2} \cdot \sqrt{3}$ must be a square root of 6. Since we have just learned that there is only one positive number which is a square root of 6, it must be true that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$.

As suggested by this example, we can prove

Theorem 11-3. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$, for any non-negative numbers a and b .

Proof: Since

$$(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2, \\ = ab,$$

it follows that the square root of ab is $\sqrt{a} \cdot \sqrt{b}$.

Problem Set 11-3a

Simplify. In those problems involving variables, indicate what restrictions must be put on the domains of the variables.

1. (a) $\sqrt{5}\sqrt{6}$ (b) $\sqrt{2}\sqrt{7}$ (c) $\sqrt{3}\sqrt{3}$
2. (a) $\sqrt{3}\sqrt{11}\sqrt{2}$ (b) $\sqrt{5}\sqrt{4 \cdot 2}$ (c) $\sqrt{3}\sqrt{12}$

3. (a) $\sqrt{2}\sqrt{x}$ (b) $\sqrt{z}\ \sqrt{3y}$ (c) $\sqrt{3}\ \sqrt{x^2}$

4. (a) $\sqrt{5}\sqrt{0}$ (b) $\sqrt{(-5)^2}\sqrt{4^3}$ (c) $\sqrt{y}\sqrt{y^3}$

5. Is it true that for every real number a, $(\sqrt{a})^2 = a$? Is it true for every non-negative number a?

6. If n is a positive integer and a, b are positive real numbers, prove that

$$\sqrt[3]{a}\ \sqrt[3]{b} = \sqrt[3]{ab}\ .$$

7. Multiply (explaining restrictions on the domains of the variables).
   (a) $\sqrt{2}\ (\sqrt{3}+\sqrt{8})$ (d) $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$
   (b) $\sqrt{x}\ (\sqrt{x}+1)$ (e) $(\sqrt{2}+\sqrt{3})^2$
   (c) $(\sqrt{a}+1)^2$ (f) $(2\sqrt{2}+1)(3\sqrt{2}-1)$

We can use the fact about square roots given in Theorem 11-3 to separate a single square root into the product of two square roots. Also, using our knowledge of prime factorization we can, for example, write

$$\sqrt{98} = \sqrt{7^2\cdot 2}.$$

We now recognize that $\sqrt{98} = \sqrt{7^2}\cdot\sqrt{2}$ (Why?)

$$= 7\sqrt{2}\ .$$

Certainly, $7\sqrt{2}$ is a simpler form than $\sqrt{98}$ because it contains no factors in the radical which are perfect squares. Recall that a phrase is called "simple" if there are no indicated operations which can be performed. Thus the indicated operation $\sqrt{7^2}$ can be performed while $\sqrt{2}$ must be left indicated.

Example 1. Simplify $\sqrt{108}$.

The prime factorization of 108 is $2^2\cdot 3^3$. Hence,

$$\sqrt{108} = \sqrt{2^2\cdot 3^3}$$

$$=\sqrt{2^2\cdot 3^2\cdot 3}$$

$$=\sqrt{2^2\cdot 3^2}\cdot\sqrt{3}$$

$$=\ 2\cdot 3\cdot\sqrt{3} = 6\sqrt{3}$$

Example 2. Simplify $\sqrt[3]{48}$.

Since $48 = 2^4\cdot 3 = 2^3\cdot 2\cdot 3$,

$$\sqrt[3]{48} = \sqrt[3]{2^3}\ \sqrt[3]{2\cdot 3}$$

$$= 2\sqrt[3]{6}\,.$$

Notice how we handled these two examples. In simplifying the *square root*, we arranged the factors so as to group the highest *even* powers of the factors. (Why even?) This insures that our new form of the expression will be simple, since we leave no indicated square roots of perfect squares. Describe the corresponding procedure for cube roots.

Example 3. Simplify $\sqrt{18x^2}$.

Again we factor: $18x^2 = 3^2\cdot x^2\cdot 2$. Hence,

$$\sqrt{18x^2} = \sqrt{3^2}\cdot\sqrt{x^2}\cdot\sqrt{2}$$

$$= 3\sqrt{x^2}\sqrt{2}.$$

Before simplifying $\sqrt{x^2}$, let us recall that $\sqrt{x^2}$ is never negative. Thus if x is positive or zero, then $\sqrt{x^2} = x$, a positive number; if x is negative then $\sqrt{x^2} = -x$, a positive number.

As a result, we see that $\sqrt{x^2} = |x|$.

Thus, $\sqrt{18x^2} = 3|x|\sqrt{2}$.

Problem Set 11-3b

In 1 - 7, simplify:

1. (a) $\sqrt{20}$ (b) $\sqrt{50}$ (c) $\sqrt{250}$ (d) $\sqrt{80}$

2. (a) $\sqrt{12}$ (b) $\sqrt{30}$ (c) $\sqrt{16}$ (d) $\sqrt{192}$

3. (a) $7\sqrt{28}$ (b) $3\sqrt{52}$ (c) $5\sqrt{13}$ (d) $4\sqrt{121}$

4. (a) $\sqrt{10}\sqrt{15}$ (b) $\sqrt{5}\sqrt{20}$ (c) $\sqrt{18}\ \sqrt{27}$

5. (a) $(6\sqrt{2})(3\sqrt{14})$ (b) $\sqrt{7}\ (2\sqrt{11})$ (c) $(3\sqrt{38})(2\sqrt{57})$

6. (a) $\sqrt{9 + 16}$ (b) $\sqrt{9 \cdot 16}$ (c) $\sqrt{9} + \sqrt{16}$

7. (a) $\sqrt[3]{16}$ (b) $\sqrt[3]{250}$ (c) $\sqrt[3]{8 + 27}$ (d) $\sqrt[3]{8}\ \sqrt[3]{27}$

8. For each of the following find a positive number x for which the sentence is true. For which ones is there a negative number which also makes the sentence true?

   (a) $x^2 = 56$ (b) $x^2 = 162$ (c) $x^3 = 56$

In 9 - 15 simplify. Indicate the domain of the variable in those problems where the domain is limited.

9. (a) $\sqrt{24x^2}$ (b) $\sqrt{24x^3}$ (c) $\sqrt{24x^5}$

10. (a) $\sqrt{32a^4}$ (b) $\sqrt[3]{32a^4}$ (c) $\sqrt[4]{32a^4}$

11. (a) $\sqrt{47}x$ (b) $\sqrt{625x^2}$ (c) $\sqrt{5x^7}$

12. (a) $\sqrt{x^4 + x^2}$ (b) $\sqrt{(x^4)(x^2)}$ (c) $\sqrt{x^4} + \sqrt{x^2}$

13. (a) $(2\sqrt{3x})(5\sqrt{6x})$ (b) $(3\sqrt{x^2y})(\sqrt{ay^2})$ (c) $\sqrt{600x} \cdot \sqrt{5000}$

14. (a) $\sqrt{16}$ (b) $\sqrt[3]{16}$ (c) $\sqrt[4]{16}$ (d) $\sqrt[5]{16}$

15. (a) $\sqrt[3]{27a^2}$ (b) $\sqrt[3]{-27b^3}$ (c) $\sqrt[3]{-27c^4}$ (d) $\sqrt[6]{2^7}$

16. Find the truth set of each of the following sentences:

    (a) $2x^2 = 32$ (b) $\frac{1}{3}y^2 = 16$ (c) $(n - 1)^2 = 9$

17. Multiply and simplify:

    (a) $3\sqrt{2}(2\sqrt{6} - \sqrt{2})$ (b) $(4\sqrt{8} - 2)2\sqrt{2}$ (c) $(\sqrt{3} - \sqrt{8})(2\sqrt{3} + \sqrt{8})$

11-4 Simplification of Radicals Involving Fractions

We have seen what can be done with integers and various powers of variables under the radical sign, and what the goals of simplifying such expressions are. What do we do if we have a fraction inside the radical?

Let us consider $\sqrt{\frac{8}{9}}$. As before, we factor to obtain $\frac{8}{9} = \frac{2^3}{3^2} = \frac{2^2}{3^2} \cdot 2.$ Then

$$\sqrt{\frac{8}{9}} = \sqrt{\frac{2^2}{3^2}} \cdot \sqrt{2}$$

$$= \frac{2}{3}\sqrt{2}\ .$$

A useful result for simplifying radicals is given by

Theorem 11-4. If $a \geq 0$ and $b > 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example.

$$\sqrt{\frac{15}{5x^2}} = \sqrt{\frac{3}{x^2}}$$

$$= \frac{\sqrt{3}}{\sqrt{x^2}}$$

$$= \frac{\sqrt{3}}{|x|}$$

For what values of x is this a real number?

Problem Set 11-4a

Simplify, indicating the domain of the variable when it is restricted.

1. (a) $\sqrt{\frac{16}{25}}$ (b) $\sqrt{\frac{3}{25}}$ (c) $\sqrt{\frac{12}{25}}$

2. (a) $\sqrt{\frac{x^2}{9}}$ (b) $\sqrt{\frac{49}{a^2}}$ (c) $\sqrt{\frac{3w^2}{y^2}}$
3. (a) $\sqrt{\frac{3x^2}{75}}$ (b) $\sqrt{\frac{4y}{9y^3}}$ (c) $\sqrt{\frac{6}{27a^2}}$
4. (a) $\frac{\sqrt{12}}{\sqrt{27}}$ (b) $\frac{\sqrt{a^5}}{\sqrt{9a^3}}$ (c) $\frac{\sqrt{xy}}{\sqrt{x^3}}$
5. (a) $\sqrt{\frac{3}{5}}\cdot\sqrt{\frac{10}{49}}$ (b) $\sqrt{\frac{m}{11}}\cdot\sqrt{\frac{3m}{44}}$ (c) $\sqrt{\frac{5}{3a}}\cdot\sqrt{\frac{3}{5a}}$
6. (a) $\frac{\sqrt{3}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}$ (b) $\frac{\sqrt{5}}{\sqrt{18}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$ (c) $\frac{\sqrt{3a^2}}{\sqrt{25x}}\cdot\frac{\sqrt{x}}{\sqrt{x}}$
7. (a) $\sqrt{1+\frac{24}{25}}$ (b) $\sqrt{\frac{35}{4}+\frac{7}{9}}$ (c) $\sqrt{\frac{15}{4}-\frac{5}{6}+\frac{5}{9}}$
8. Prove Theorem 11-4.

We come now to the case of radicals containing fractions whose denominators are not perfect squares. What do we propose to do with $\sqrt{\frac{3}{5}}$, for example? We know that

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}}\,. \quad \text{(Why?)}$$

In this form, $\frac{\sqrt{3}}{\sqrt{5}}$ involves two square roots of integers, and this certainly is not as simple as if it involved only one. How should we change $\frac{\sqrt{3}}{\sqrt{5}}$ so that there would be only one radical (with an integer under the radical sign) in the whole expression? We have two choices: We must somehow get rid of either the $\sqrt{3}$ or the $\sqrt{5}$. How, for instance, might we get rid of $\sqrt{3}$? If we were to multiply the whole expression by $\frac{\sqrt{3}}{\sqrt{3}}$, which is another way of saying 1, then in the numerator we would have $\sqrt{3}\cdot\sqrt{3}$, which is just 3, and the radical would be gone. In the denominator we would have $\sqrt{5}\cdot\sqrt{3}$, which is $\sqrt{15}$, and this we can simplify no more, since 15 contains no perfect square factors.

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}\,\sqrt{3}}{\sqrt{5}\,\sqrt{3}} = \frac{3}{\sqrt{15}}$$

(Justify each step.)

We did remark, however, that we had another choice. We could get rid of the $\sqrt{5}$ instead.
In this case (justify each step),

$$\sqrt{\frac{3}{5}} = \frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{3}\,\sqrt{5}}{\sqrt{5}\,\sqrt{5}} = \frac{\sqrt{15}}{5} .$$

Which of our two final expressions would we prefer? Each of them contains $\sqrt{15}$ and no other radicals. If you had a decimal approximation to $\sqrt{15}$, such as 3.873, how would you find a numerical approximation to $\sqrt{\frac{3}{5}}$? The form $\frac{3}{\sqrt{15}}$ would leave you with the problem of computing $\frac{3}{3.873}$; the form $\frac{\sqrt{15}}{5}$ leads to $\frac{3.873}{5}$. Which is easier?

For this reason, then, the form which leaves the denominator rational is often preferable to the form which leaves the numerator rational. Quite naturally, the process which leads to a rational denominator is called "rationalizing the denominator".

Example 1. Rationalize the denominator of

(a) $\sqrt{\frac{7}{12}}$, (b) $\sqrt{\frac{3}{2x^2}}$, $x \neq 0$.

(a) $\sqrt{\frac{7}{12}} = \frac{\sqrt{7}}{2\sqrt{3}} = \frac{\sqrt{7}}{2\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{6}$

(b) $\sqrt{\frac{3}{2x^2}} = \frac{\sqrt{3}}{\sqrt{2}\,|x|} = \frac{\sqrt{3}}{\sqrt{2}\,|x|}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2|x|}$ $x \neq 0$.

Example 2. Rationalize the numerator of

$\sqrt{\frac{x}{2}}$, $x \geq 0$.

$$\sqrt{\frac{x}{2}} = \frac{\sqrt{x}}{\sqrt{2}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{x}{\sqrt{2x}}, \quad x > 0.$$

(Why is it necessary in the original problem to restrict x to non-negative values? Why must we restrict x in the rationalized form to positive values?)

Problem Set 11-4b

Rationalize the denominator. Indicate the domain of the variable in each case where it is restricted.

1. (a) $\sqrt{\frac{1}{3}}$ (b) $\sqrt{\frac{1}{4}}$ (c) $\sqrt{\frac{1}{12}}$ (d) $\sqrt{\frac{1}{27}}$
2. (a) $\sqrt{\frac{9}{50}}$ (b) $\sqrt{\frac{5}{18}}$ (c) $3\sqrt{\frac{7}{36}}$ (d) $\frac{1}{5}\sqrt{\frac{75}{63}}$
3. (a) $\frac{\sqrt{15}}{\sqrt{30}}$ (b) $\frac{\sqrt{7}}{2}$ (c) $\frac{\sqrt{240}}{\sqrt{24}}$ (d) $-\frac{9\sqrt{5}}{12\sqrt{15}}$
4. (a) $\sqrt{\frac{3b}{5}}$ (b) $\sqrt{\frac{2a}{7b}}$ (c) $\sqrt{\frac{2x^2}{9}}$ (d) $\sqrt{\frac{5}{x^3}}$
5. (a) $\sqrt[3]{\frac{1}{2}}$ (b) $\sqrt[3]{\frac{1}{9}}$ (c) $\sqrt[3]{\frac{4}{a^2}}$ (d) $\sqrt[3]{\frac{5}{4a}}$
6. (a) $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{2}{15}}$ (b) $\sqrt{\frac{2a}{45}} \cdot \sqrt{\frac{a^5}{2}}$ (c) $3\sqrt{\frac{1}{3}} \cdot 5\sqrt{\frac{3}{5}}$
7. (a) $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{6}}$ (b) $\frac{\sqrt{6} + \sqrt{24}}{\sqrt{6}}$ (c) $\sqrt{\frac{9 + 16}{25}}$
8. Rationalize the numerator.

   (a) $\frac{\sqrt{x}}{x}$ (b) $\frac{\sqrt{b}}{1 + \sqrt{b}}$ (c) $\frac{3\sqrt{7}}{14\sqrt{9}}$
9. Simplify:

   (a) $(\sqrt{3} + \sqrt{2})^2$ (b) $(\sqrt{x} + 1)^2$ (c) $\left(\sqrt{a} + \sqrt{\frac{1}{a}}\right)^2$

Having seen what can be done to products and quotients of radicals in order to simplify them, we turn next to sums and differences of radicals. We are always guided by the agreement that a phrase is not in simple form unless it is free of indicated operations which can be performed. Consider $\sqrt{2} + \sqrt{3}$. Certainly, $\sqrt{2}$ and $\sqrt{3}$ are each in simplified form and we cannot perform the indicated operation of adding $\sqrt{2}$ and $\sqrt{3}$. Hence, $\sqrt{2} + \sqrt{3}$ is a simple phrase.

On the other hand, consider $4\sqrt{3} - 3\sqrt{12}$. By the familiar procedure, we obtain:

$$4\sqrt{3} - 3\sqrt{12} = 4\sqrt{3} - 6\sqrt{3} = (4 - 6)\sqrt{3} = -2\sqrt{3}.$$

The last equality follows from the distributive property. Thus we were able to simplify considerably in this case.

What is the difference between these two examples? If you have a sum of different square roots, no one of which contains a perfect square factor, it is in simple form. If one or more of the square roots in the sum does contain a perfect square factor, there is a possibility of simplification by the distributive property.

Problem Set 11-4c

Simplify:

1. (a) $\sqrt{2} + \sqrt{8}$ (b) $\sqrt{18} - \sqrt{27}$ (c) $2\sqrt{12} + 3\sqrt{75}$

2. (a) $8\sqrt{\frac{1}{2}} - \frac{1}{2}\sqrt{8}$ (b) $\sqrt{\frac{5}{9}} + \sqrt{\frac{9}{5}}$ (c) $\frac{1}{3}\sqrt{63} + 7\sqrt{3}$

3. (a) $\sqrt{34} + \frac{1}{2}\sqrt{16} - \sqrt{20}$ (b) $\frac{1}{4}\sqrt{288} - \frac{1}{6}\sqrt{72} + \frac{1}{\sqrt{24}}$

4. (a) $\sqrt[3]{6} + \sqrt{6}$ (b) $\sqrt[3]{6} + \sqrt[3]{48}$ (c) $\sqrt[4]{32} + 3\sqrt[4]{1250}$

5. Simplify, assuming that a and b are positive numbers.

   (a) $\sqrt{9a} + \sqrt{4a}$ (b) $a\sqrt{3a} + 2\sqrt{a^3}$ (c) $\frac{\sqrt{a}}{\sqrt{b}} - \frac{\sqrt{ab}}{b}$

6. For what values of x are the following sentences true?

   (a) $x^2 = 5$ (b) $\sqrt{x} = 5$ (c) $\sqrt{x^2} = 5$

11-5. Square Roots

You have learned to recognize instantly the square roots of 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121 and 144. You may even be able to identify the square roots of 225, 400, 625, 900, etc., because the square roots of these numbers are rational. On the other hand, what is the square root of 3? Of .0621? These are entirely different questions, because $\sqrt{3}$ and $\sqrt{.0621}$ are irrational numbers.

You are able to locate the number $\sqrt{49}$ on the number line simply by locating 7. But how can you locate the irrational number $\sqrt{3}$? In fact, what is meant by "finding" or "locating" or "evaluating" an irrational number such as $\sqrt{3}$? We mean by "evaluating $\sqrt{3}$" the process of finding two rational numbers such that $\sqrt{3}$ lies between them. Each of these rational numbers is called an approximation to $\sqrt{3}$. The closer the two rational numbers are to each other, the better the approximation.

Thus, we evaluate an irrational number in the sense of finding rational numbers which are close to the irrational number.

Since $1^2 = 1$, $(\sqrt{3})^2 = 3$, and $2^2 = 4$, and since $1 < 3 < 4$ we know that

$$1 < \sqrt{3} < 2 .$$

These approximations to $\sqrt{3}$ are the nearest integers. Next we consider the squares:

$$(1.1)^2 = 1.21, \quad (1.2)^2 = 1.44, \quad (1.3)^2 = 1.69,$$
$$(1.4)^2 = 1.96, \quad (1.5)^2 = 2.25, \quad (1.6)^2 = 2.76,$$
$$(1.7)^2 = 2.89, \quad (1.8)^2 = 3.24, \quad (1.9)^2 = 3.61.$$

A glance shows us that

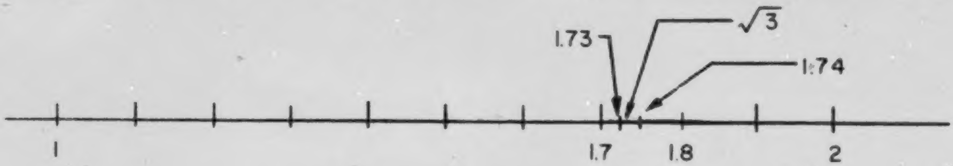
$$1.7 < \sqrt{3} < 1.8.$$

These approximations are to the nearest tenths. Again, trying the squares of the hundredths between 1.7 and 1.8 we find that $(1.73)^2 = 2.9929$ and $(1.74)^2 = 3.0286$. Hence,

$$1.73 < \sqrt{3} < 1.74,$$

and we have approximations to $\sqrt{3}$ which are the nearest hundredths. Write the squares of thousandths between 1.73 and 1.74 and find the two thousandths between which $\sqrt{3}$ lies. (Hint: try 1.732.) This process could be continued without end, each time finding two rational numbers, closer and closer together, with $\sqrt{3}$ lying between them. We say that each of these successive approximations contains one more digit than the preceding.

You see that we can squeeze $\sqrt{3}$ between two arbitrarily close rational numbers. This can be done for any irrational number,



although the process we used here is very slow and laborious.

Fortunately, there are more efficient ways to evaluate square roots than by the trial and error method we used. We shall describe one such method. First, we make a rough estimate for a first approximation. Second, we refine our first approximation by a special procedure.

In order to make a rough estimate of the value of the square root of a number, we should first put the number in a standard form. By this we mean write the number as the product of a number a between 1 and 100 and a power of 10. For example,

$$392 = 39.2 \times 10 = 3.92 \times 10^2$$
$$3920 = 39.2 \times 10^2 = 3.92 \times 10^3$$
$$39200 = 39.2 \times 10^3 = 3.92 \times 10^4$$

We can see, for instance, that we obtain 39.2×10^2 by multiplying 3920 by $10^{-2} \times 10^2$. (What number is $10^{-2} \times 10^2$?)

$$3920 = 3920(10^{-2} \times 10^2) = (3920 \times 10^{-2}) \times 10^2 = 39.2 \times 10^2$$

Notice that each of these numbers can be written either as 39.2 times a power of ten or as 3.92 times a power of 10.

Similarly, we may write

$$0.392 = 39.2 \times 10^{-2} = 3.92 \times 10^{-1}$$

$$0.0392 = 39.2 \times 10^{-3} = 3.92 \times 10^{-2}$$

$$0.00392 = 39.2 \times 10^{-4} = 3.92 \times 10^{-3},$$

where we obtained 39.2×10^{-4}, for instance, by multiplying 0.00392 by $10^4 \times 10^{-4}$.

$$0.00392 = 0.00392(10^4 \times 10^{-4}) = (0.00392 \times 10^4) \times 10^{-4}$$
$$= 39.2 \times 10^{-4}.$$

You see that any number can be written in two forms as

$$a \times 10^k,$$

where a is a number between 1 and 100 and k is an integer. In one of the forms, k is an even integer; in the other, k is odd. Write 29300 in two standard forms. Do the same for 0.000293. For 0.00293. For 2.93.

Now we are ready to make a rough estimate of the square root of a number.

Our method of estimating will depend on writing the number as a product and using the theorem $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for any non-negative numbers a and b. We shall choose a and b in such a way that for each of them it is easy to estimate a square root. In particular, if we choose b as 10^k where k is an even integer, we can find $\sqrt{b}$ exactly. Find the powers of 10 which complete the following sentences:

$\sqrt{10^6} = 10^3$ $\qquad$ $\sqrt{10^{-2}} = ?$

$\sqrt{10^2} = ?$ $\qquad$ $\sqrt{10^{-8}} = ?$

$\sqrt{10^4} = ?$ $\qquad$ $\sqrt{10^0} = ?$

$\sqrt{10^{14}} = ?$ $\qquad$ $\sqrt{10^{-6}} = ?$

If we write our number in the standard form described above, the first factor will be a number between 1 and 100. For instance $354000 = 35.4 \times 10^4$. Thus

$$\sqrt{354000} = \sqrt{35.4}\sqrt{10^4} .$$

For 35.4 and for any number between 1 and 100 it is not hard to find the integers which come closest to being the square root of the number. This is possible because we are familiar with all the numbers between 1 and 100 which are perfect squares. Thus 35.4 is between 25 and 36; hence, $\sqrt{35.4}$ is between 5 and 6.

For each of the following, find two successive integers between which the irrational number lies.

$\sqrt{19}$	$\sqrt{96}$	$\sqrt{1.38}$
$\sqrt{54}$	$\sqrt{11.6}$	$\sqrt{7}$
$\sqrt{5}$	$\sqrt{79.42}$	$\sqrt{30.2}$

Now let us combine the ideas we have developed and use them to estimate $\sqrt{354000}$.

$$\sqrt{354000} = \sqrt{35.4}\sqrt{10^4}$$

Since $5 < \sqrt{35.4} < 6,$

then,

$$5 \times 10^2 < \sqrt{35.4}\sqrt{10^4} < 6 \times 10^2,$$

$$500 < \sqrt{354000} < 600 .$$

Since $\sqrt{35.4}$ is closer to 6 than to 5, we write

$$\sqrt{354000} \approx 600.$$

Here we are using the symbol "$\approx$" to mean "is approximately equal to". How could you verify that 600 is approximately equal to $\sqrt{354000}$? Is 600^2 somewhat near 354000?

Example 1. Estimate $\sqrt{.00537}$.

$$\sqrt{.00537} = \sqrt{53.7}\sqrt{10^{-4}}$$

Since $$7 < \sqrt{53.7} < 8,$$

then,
$$7 \times 10^{-2} < \sqrt{53.7}\sqrt{10^{-4}} < 8 \times 10^{-2},$$
$$.07 < \sqrt{.00537} < .08.$$

Since $\sqrt{53.7}$ is closer to 7 than to 8, we write
$$\sqrt{.00537} \approx .07.$$

Example 2. Estimate $\sqrt{546}$

$$\begin{aligned}\sqrt{546} &= \sqrt{5.46} \times \sqrt{10^2}\\ &\approx 2 \times 10\\ \sqrt{546} &\approx 20\end{aligned}$$

Problem Set 11-5a

Find a first approximation to each of the following.

1. (a) $\sqrt{27}$ (b) $\sqrt{67}$ (c) $\sqrt{17}$ (d) $\sqrt{47}$ (e) $\sqrt{7}$
2. (a) $\sqrt{7900}$ (b) $\sqrt{.79}$ (c) $\sqrt{.000079}$ (d) $\sqrt{79}$
3. (a) $\sqrt{846}$ (b) $\sqrt{84600}$ (c) $\sqrt{.0846}$ (d) $\sqrt{84.6}$
4. (a) $\sqrt{2280000000}$ (b) $\sqrt{.000000053}$
5. (a) $\sqrt{19 \times 10^{-34}}$ (b) $\sqrt{77 \times 10^{16}}$

The next step is to derive a better approximation from the first rough approximation. In order to find a better approximation it is sufficient to work on only the first of the two factors obtained in our procedure. (Why?)

Let us return to Example 1 above in which we estimate that 7 is the nearest integer to $\sqrt{53.7}$.

Consider the product $pq = 53.7$. If $p = q$, then p (and q) is $\sqrt{53.7}$. If p is larger than $\sqrt{53.7}$, then in order that pq still may be equal to 53.7, q must be smaller than $\sqrt{53.7}$*. We see that $\sqrt{53.7}$ is then between p and q. Similarly if p is too small, q is too large, and the square root is still between p and q.

In our case, let $p = 7$; then the value of q such that $pq = 53.7$ is $\frac{53.7}{7}$, which is approximately 7.68. According to our discussion above, $\sqrt{53.7}$ is between 7 and 7.68. We take as our next approximation a number half way between them. We therefore find the average of p and q, namely $\frac{p+q}{2}$.

In this case $\frac{7 + 7.68}{2} = \frac{14.68}{2} = 7.34$.

It can be shown that this new number, 7.34, is considerably closer to $\sqrt{53.7}$ than our first guess, 7, and is greater than $\sqrt{53.7}$. Thus, $7.32 < \sqrt{53.7} < 7.34$. Check this by squaring 7.32 and 7.34.

If we need a still closer approximation, we can use this second approximation, 7.34, as p, compute $q = \frac{53.7}{p}$, and find the average of the new p and q. To simplify the computation let us round 7.34 to 7.3.

$$p = 7.3$$

$$q = \frac{53.7}{7.3} \approx 7.356$$

$$\frac{p+q}{2} \approx \frac{7.3 + 7.356}{2}$$

---

* If $pq = 53.7$ and $p > \sqrt{53.7}$, then

$$pq > \sqrt{53.7}\,q,$$

$$53.7 > \sqrt{53.7}\,q,$$

$$\sqrt{53.7} > q.$$

$$\approx \frac{14.656}{2}$$

$$\approx 7.328$$

$$\sqrt{53.7} \approx 7.328$$

Hence $\sqrt{.00537} \approx 7.328 \times 10^{-2}$,

$$\sqrt{.00537} \approx .07328.$$

Thus, $.07327 < \sqrt{.00537} < .07328$.

Perhaps you would like to see how close this approximation is by squaring .07328.

Example 1. Find a second approximation to $\sqrt{763}$.

$$\sqrt{763} = \sqrt{7.63} \times \sqrt{10^2}$$

$$\approx 3 \times 10$$

estimate	divide	average
p	$q = \frac{7.63}{p}$	$\frac{p+q}{2}$
3	2.54	2.77

$$\frac{3 + 2.54}{2} = \frac{5.54}{2} = 2.77$$

$$\sqrt{763} \approx 2.77 \times 10$$

$\sqrt{763} \approx 27.7$ and $27.6 < \sqrt{763} < 27.7$.

Example 2. Find a third approximation to $\sqrt{.2138}$.

$$\sqrt{.2138} = \sqrt{21.38} \times \sqrt{10^{-2}}$$

$$\approx 5 \times 10^{-1}$$

estimate	divide	average
p	$q = \frac{21.38}{p}$	$\frac{p+q}{2}$
5	4.28	4.64
4.6	4.648	4.624

$$\frac{5 + 4.28}{2} = \frac{9.28}{2} = 4.64$$

$$\frac{4.6 + 4.648}{2} = \frac{9.248}{2}$$

$$= 4.624$$

Hence $\sqrt{.2138} \approx 4.624 \times 10^{-1}$

$\sqrt{.2138} \approx .4624$ and $.4623 < \sqrt{.2138} < .4624$

Summary: To approximate $\sqrt{x}$, first write it as the square root of a number a between 1 and 100 times an even power of 10:

$$\sqrt{x} = \sqrt{a}\ \sqrt{10^{2n}}$$

Second, find an integer p between 1 and 10 inclusive which is the first approximation to $\sqrt{a}$. To find the second approximation, divide a by p to find q, $(q = \frac{a}{p})$, and determine the average of p and q. This average, $\frac{p+q}{2}$, is the second approximation to $\sqrt{a}$.

Then $\sqrt{x} \approx \frac{p+q}{2} \times 10^n$.

Carry out the division $\frac{a}{p}$ to three digits and remember that the third digit may be in error. If more accuracy is wanted, round the second approximation, $\frac{p+q}{2}$, to two digits and repeat the process of dividing and averaging. The resulting third approximation will usually be in error by no more than 1 or 2 in the fourth digit. By this we mean that the difference between $\sqrt{a}$ and its approximation is less than .002. If still more accuracy is desired, repeat the process of dividing and averaging (but do not round off the divisor).

Each new approximation is larger than the value of the square root, and each will have roughly twice as many correct digits as the preceding approximation.

How do we know these statements are true in general? The above numerical examples serve to illustrate a result which can be proved by applying the basic properties of the operations.

Problem Set 11-5b

1. Find the second approximations to:

(a) $\sqrt{796}$ (c) $\sqrt{0.0884}$ (e) $\sqrt{0.00580}$

(b) $\sqrt{73}$ (d) $\sqrt{304000}$ (f) $\sqrt{9999900}$

2. Find the third approximations to:

(a) $\sqrt{0.00470}$ (d) $\sqrt{3.1416}$

(b) $\sqrt{0.273}$ (e) $\sqrt{70260}$

(c) $\sqrt{5280}$ (f) $\sqrt{502060}$

3. A table of decimal approximations to the square roots of integers from 1 to 100 gives the following:

$$\sqrt{72} \approx 8.485$$

$$\sqrt{8} \approx 2.828.$$

Using these approximations, find an approximation to:

(a) $\sqrt{0.0072}$ (d) $\sqrt{0.08}$

(b) $\sqrt{720000}$ (e) $\sqrt{800}$

(c) $\sqrt{.72}$ (f) $\sqrt{8,000,000}$

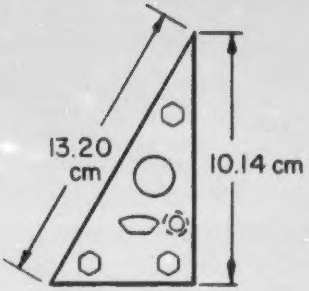
4. Find second approximations to the elements of the truth set of each of the following:

(a) $x^2 = 0.0124$ (b) $x^2 - 519 = 0$

5. The students attending Lincoln High School have a habit of cutting across a vacant lot near the school instead of following the sidewalk around the corner. The lot is a rectangular lot 200 feet by 300 feet, and the short-cut follows a straight line from one corner of the lot to the opposite corner. How long, to the nearest foot, is the short-cut?

6. A fourteen foot ladder rests against a vertical wall, the foot of the ladder being seven feet from the base of the wall. Determine the height at which the ladder touches the wall. How close an approximation would be reasonable to expect in this case?

7. A machine part is in the shape of a right triangle having the hypotenuse 13.20 cm. long and one leg 10.14 cm. long. Find the other leg to the third approximation.



Review Problems

1. Simplify, indicating the domain of the variable when restricted.

(a) $\sqrt{12}$

(b) $\frac{1}{\sqrt{36}}$

(c) $\sqrt{8a}$

(d) $\sqrt{\frac{16}{3}}$

(e) $\sqrt{18x^2}$

(f) $\sqrt{6}\sqrt{24}$

(g) $\sqrt{8} - \sqrt{18}$

(h) $(a^2bc)(ab^2c)$

(i) $\sqrt{2}(\sqrt{2} + \sqrt{6})$

2. Simplify, indicating the domain of the variable when restricted.

(a) $\sqrt{48} - \sqrt{75} + \sqrt{12}$

(b) $\sqrt{5}\sqrt{20}$

(c) $\sqrt{4(a + b)^2}$

(d) $(3^a)(2^b)$

(e) $\sqrt{\frac{1}{2}} - \frac{\sqrt{9}}{\sqrt{8}}$

(f) $\sqrt{2}(\sqrt{2} - \sqrt{6})$

(g) $\sqrt[3]{54}$

(h) $-\sqrt{32x^3y}$

(i) $\frac{\sqrt{3}}{\sqrt{2}} + 1$

3. Simplify, indicating the domain of the variable when restricted.

(a) $2\sqrt{12a^2} - \frac{3|a|}{\sqrt{3}} - \frac{1}{4}\sqrt{48a^2}$

(b) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$

(c) $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

(d) $\sqrt{\frac{4m^2}{q}} + \sqrt{98m^2q^3}$

(e) $\sqrt{\frac{2}{3}}\sqrt{\frac{5}{6}}\sqrt{\frac{3}{2}}$

(f) $\frac{1}{\sqrt[3]{250x}}$

(g) $\sqrt{3p}\sqrt{6p^3}$

(h) $\sqrt[3]{\frac{1}{4a}} - \sqrt[3]{16a^5}$

(i) $\sqrt{4a^2 + 4b^2}$

4. Solve the following:

(a) $\sqrt{x} = 2$ (d) $m^2 \leq 16$

(b) $\sqrt[3]{n} = 4$ (e) $3 = \sqrt{t + 1}$

(c) $y^2 = 2$ (f) $2|x| + \sqrt{x^2} = 3$

5. In each of the following use one of the symbols $<$, $=$, $>$ between the two given phrases so as to make a true sentence.

(a) $\frac{1}{x} + \frac{2}{3x}$, $\frac{1}{3}$, for $x = 5$

(b) $x + \sqrt{2}$, $\sqrt{2}$, for $x > 0$

(c) $\sqrt{a^2 + b^2}$, $(a + b)$, for $a > 0$ and $b > 0$

(d) $\sqrt[4]{3^2}$, $\sqrt{3}$

(e) $(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n})$, $(m - n)$, for $m > 0$ and $n > 0$

(f) $(|x| + 5\sqrt[3]{125})$, $(-x^3)$, for $x = 25$

(g) $\sqrt{\frac{3^2 \cdot 5^2}{256}}$, $7\sqrt{\frac{1}{8}}$

6. Evaluate $\sqrt{390}$ (to the second approximation).

7. Evaluate $\sqrt{3900}$ (to the third approximation).

8. Express as powers of simple numerals, if possible.

(a) $3^2 \cdot 3^3$ (d) $3^3 + 3^3 + 3^3$

(b) $3^2 \cdot 2^2$ (e) $3^4 + 3^4 + 3^4$

(c) $3^2 \cdot 2^3$ (f) $3^2 + 2^2$

9. Express as powers of 10. (n is an integer.)

(a) $10^{-2} \times 10^2$ (c) $10^n \times 10^2$ (e) $10^{-3} \times 10^{-5} \times 10^3$

(b) $\frac{10^3 \times 10^{-1}}{10^1}$ (d) $\frac{10^5 \times 10^{-3}}{10^{-2}}$ (f) $(10^{2n})^3$

10. Simplify: (assume no variable takes on the value zero.)

(a) $\dfrac{\frac{2}{3}}{\frac{5}{6}}$ (d) $\dfrac{\frac{1}{4} + \frac{1}{3}}{\frac{5}{12}}$

(b) $\dfrac{\frac{b^3}{2}}{\frac{b}{6}}$ (e) $\dfrac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

(c) $\dfrac{\frac{mn}{q^2}}{\frac{2nq}{m^2}}$ (f) $\dfrac{\frac{x + 3}{2}}{\frac{x(x + 3)}{3}}$

11. Solve:

(a) $\frac{1}{3}x - 1 > \frac{1}{2}x$ (d) $|m| - \frac{3}{20} = \frac{1}{5}|m|$

(b) $\frac{8 - x}{3} = \frac{10x}{6}$ (e) $(q + 3)^2 = q^2 + 21$

(c) $\frac{y}{3} + \frac{y}{5} < 1$ (f) $\frac{x - 3}{x + 2} = \frac{x - 2}{x + 1}$

12. A remarkable expression which produces many primes is

$$n^2 - n + 41.$$

If n is any number of the set $\{1, 2, 3, \ldots, 40\}$ the value of the expression is a prime number, but for $n = 41$ the expression fails to give a prime number. Tell why it fails. If an algebraic sentence is true for the first 400 values of the variable, is it then necessarily true for the 401st?

13. A procedure sometimes used to save time in averaging large numbers is to guess at an average, average the differences, and add that average to your guess. Thus, if the numbers to be averaged -- say your test scores -- are 78, 80, 76, 72, 85, 70, 90, a reasonable guess for your average might be 80. We find how far each of our numbers is from 80.

$78 - 80 = -2$
$80 - 80 = 0$
$76 - 80 = -4$
$72 - 80 = -8$
$85 - 80 = 5$
$70 - 80 = -10$
$90 - 80 = 10$

The sum of the differences is -9. The average of the differences is $-\frac{9}{7}$. Adding this to 80 gives $78\frac{5}{7}$ for the desired average. Can you explain why this works?

The weights of a university football team were posted as 195, 205, 212, 201, 198, 232, 189, 178, 196, 204, 182. Find the average weight for the team by the above method.

14. A rat which weighs x grams is fed a rich diet and gains 25% in weight. He is then put on a poor diet and loses 25% of his weight. Find the number of grams difference in the weight of the rat from the beginning of the experiment to the end.

15. A man needs 7 gallons of paint to paint his house. He bought three times as much grey paint at \$6 a gallon as white paint at \$7 a gallon. If his paint bill was less than \$50 how many gallons of each color paint did he buy? (Assume the smallest size paint can available is the quart size.)

16. Prove: If $a > 0$, $b > 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$.
    Hint: Use the comparison property in an indirect proof.

Chapter 12

POLYNOMIAL AND RATIONAL EXPRESSIONS

12-1. Polynomials and Factoring

In Chapter 10 we found that there are many advantages to having a numeral in factored form. Consider, for example, the number 288. The common name for this number is actually an abbreviation of "2(100) + 8(10) + 8". This is the form on which most of the arithmetic involving the number is based. We also have the factored form "$2^5 \cdot 3^2$". If you want to tell whether or not 288 is a perfect square, which form would you use? What if you wanted to find the simplest form for $\sqrt{288}$? In algebra, the factored form for a positive integer is frequently the most convenient.

Since the prime factored form for integers has turned out to be so useful, it is natural to ask whether we can similarly write algebraic expressions in "factored form," that is, as indicated products of simpler phrases. You have already done problems of this kind. What properties of the real numbers enable us to write, for any real number x,

$$3x^2 + x = x(3x + 1)?$$

We could also write $3x^2 + x$ in the factored form

$$3x^2 + x = (x^2 + 1) \cdot \frac{3x^2 + x}{x^2 + 1}.$$

Why is this latter form not as interesting as the first? One reason is that the factor $\frac{"3x^2 + x"}{x^2 + 1}$ is more complicated than the given expression; it involves division, while "$3x^2 + x$" involves only addition and multiplication. We are reminded of our study of positive integers; it was useful to factor positive integers over the positive integers, but not over the rationals or reals.

What type of expression should correspond here to a positive integer? In other words, for what types of expressions will the problem of factoring be interesting? Certainly, phrases such as $3x^2 + x$, x and $3x + 1$ must be included, while phrases such as $\frac{3x^2 + x}{x^2 + 1}$ should be excluded.

Let us look more closely at the form of the phrase

$$3x^2 + x .$$

It involves the integer 3, the variable x, and indicated operations of addition and multiplication. On the other hand, the phrase

$$\frac{3x^2 + x}{x^2 + 1}$$

involves the integers 3 and 1, the variable x, and indicated operations of addition, multiplication and division. As we have seen, the essential difference between these phrases is that the second involves division while the first does not.

Thus, we are led to a general definition of phrases such as "$3x^2 + x$ ".

> A phrase formed from integers and variables, with no indicated operations other than addition, subtraction, multiplication or taking opposites, is called a polynomial over the integers.

If there is just one variable involved, say x, we have a polynomial in x. Thus, "$3x^2 + x$ ", "$3x + 1$ ", "x" are polynomials in x over the integers.

Later we will extend our study to polynomials over the real numbers, but for the time being "polynomial" will mean "polynomial over the integers".

Problem Set 12-1a

In Problems 1 and 2, which are polynomials over the integers?

1. (a) $3t + 1$
   (b) $t + \frac{1}{3}$
   (c) $3a^2b$
   (d) 2
   (e) $(x - 2)(x + 3)$
   (f) $\frac{3}{2}$
   (g) $rq - \sqrt{2}$
   (h) $|x| + 1$

2. (a) $(s + 5)(t - 1)u$
   (b) $\frac{2}{3}(x - 4)$
   (c) $2(x - 1) + x(x - 1)$
   (d) $(3x - y + 7)^3$
   (e) $\frac{x + 1}{x - 1}$
   (f) $\frac{x + y}{2}$
   (g) $\frac{x^2}{a^2} + 2\frac{x}{a} + 1$
   (h) $\frac{3s(u + v)}{s}$

In Problems 3 and 4, simplify by performing the indicated multiplications and collecting terms. Is the result always a polynomial over the integers?

3. (a) $2x(x - 2)$
   (b) $xy(x - 2y)$
   (c) $(t - 2)(t + 3)$
   (d) $(-3xy^2)(\frac{3}{11}x^2y)z$
   (e) $(u + \frac{1}{2})(u - \frac{1}{2})$
   (f) $(x + 2)(x + 2)$
   (g) $(3t - 8)(6t + 11)$
   (h) $2(y - 1) + y(y - 1)$

4. (a) $(s^2 - 1) - (s + 1)(s - 1)$
   (b) $(a + \sqrt{2})(a - \sqrt{2})$
   (c) $3s(u + v)$
   (d) $3u(u + \frac{1}{3}) - uv$
   (e) $2(s + 1) - 6st$
   (f) $(x - 2y + 1)(2x + y - 2)$

5. Will indicated sums and products of polynomials over the integers always be polynomials over the integers?

6. Can an indicated quotient of two polynomials ever be a polynomial? Can such a quotient ever be simplified to be a polynomial? Give an example.

Let us return to the problem of factoring expressions which led us in the first place to consider polynomials. Just as the problem of factoring numbers was most interesting when it was restricted to positive integers, so the problem of factoring expressions is most interesting when it is restricted to polynomials.

Recall the expression "$3x^2 + x$" which we considered at the beginning of this section. This is a polynomial over the integers, and we saw that the distributive property could be used to write it in the factored form

$$3x^2 + x = x(3x + 1)$$

Since "x" and "$3x + 1$" are also polynomials over the integers, we say that we have factored a polynomial over the integers.

This suggests the reason for our dislike of the factorization

$$3x^2 + x = (x^2 + 1)\frac{3x^2 + x}{x^2 + 1} .$$

We want the factors of "$3x^2 + x$" to be the same kind of phrases as "$3x^2 + x$", namely, polynomials. Thus

> The problem of factoring is to write a given polynomial as an indicated product of polynomials.

Just as in the case of positive integers, we also wish to carry the factoring process for polynomials as far as possible, namely, until the factors obtained cannot be factored into "simpler" polynomials.

Factoring can be thought of as the inverse process of what we have called "simplification". For example, given the polynomial

$$x(3x + 5y)(2y - x),$$

we "simplify" it by performing the indicated multiplications and collecting terms, thus obtaining the polynomial

$$-3x^3 + x^2y + 10xy^2.$$

On the other hand, in order to factor the polynomial

$$-3x^3 + x^2y - 10xy^2,$$

we must somehow reverse the simplification steps so as to obtain

$$x(3x + 5y)(2y - x).$$

By examining carefully the process of simplifying indicated products, we shall work out in this chapter techniques for handling problems of this kind.

Notice that the polynomial obtained in the above simplification is a sum of terms, each of which is also a polynomial. A polynomial which involves at most the taking of opposites and indicated products is called a monomial. Hence, each of the terms $-3x^3$, x^2y, $-10xy^2$ is a monomial, and we have written the given polynomial as a sum of monomials. Any polynomial can be written in this way as a sum of monomials.

When a polynomial in one variable is written as a sum of monomials, we say its degree is the highest power of the variable in any monomial. Thus, for example,

$$3x^2 - 2x + 4$$

is a polynomial of degree two. We also say that "3" is the coefficient of x^2, "-2" is the coefficient of x, and "4" is the constant. A polynomial of degree two is called a quadratic polynomial.

In factoring polynomials in one variable, our objective is to obtain polynomial factors of lowest possible degree.

Problem Set 12-1b

In Problems 1, 2, and 3, indicate which are examples of factoring polynomials over the integers.

1. (a) $ax + 2ay = a(x + 2y)$

   (b) $x^2 - 4 = (x + 2)(x - 2)$

   (c) $s^2 - 3 = (s + \sqrt{3})(s - \sqrt{3})$

   (d) $3t - 5 = 3(t - \frac{5}{3})$

   (e) $a^2 - 2a + 1 = (a - 1)^2$

   (f) $9y^2 - 4 = (3y + 2)(3y - 2)$

2. (a) $u - 9 = (\sqrt{u} + 3)(\sqrt{u} - 3)$

   (b) $r^3 - 3s^2 = (t^2 + 1)\frac{r^3 - 3s^2}{t^2 + 1}$

   (c) $3x^3y^2 - 2x^2y^3 = (3x - 2y)x^2y^2$

   (d) $a(c + d) - b(c + d) = (a - b)(c + d)$

   (e) $t^2 - 1 = (|t| + 1)(|t| - 1)$

   (f) $x^2 + 5x + 6 = (x + 2)(x + 3)$

3. (a) $3x(y^2 + 1) - 2z(y^2 + 1) = (y^2 + 1)(3x - 2z)$

   (b) $h^2 - h + \frac{1}{4} = (h - \frac{1}{2})^2$

   (c) $3rs + 5r - 10st - 6t = r(3s + 5) - 2t(5s + 3)$

   (d) $ac + ad - bc - bd = a(c + d) - b(c + d)$

   (e) $a^2x^2 + ax + 1 = a^2\left(x^2 + (\frac{x}{a}) + (\frac{1}{a})^2 \right)$

4. Verify that the factorings in Problems 1, 2, and 3 are correct performing the indicated multiplication on the right in each case.

5. Which of the following polynomials are in factored form?

   (a) $(x - 3)(x - 2)$

   (b) $(x - 3) + (x - 2)$

   (c) $(x - 3)\ x - 2$

   (d) $(x - 3)(x - 2)(x - 1) + (x - 1)$

   (e) $(x + y + z)(x - y - z)$

   (f) $3z(z + 1) - 2z$

In Problems 6, 7, and 8, use the distributive property, if possible, to factor as completely as you can each of the following polynomials.

6. (a) $a^2 + 2ab$

   (b) $3t - 6$

   (c) $ab + ac$

   (d) $3x(xz - yz)$

   (e) $ax - ay$

   (f) $6p - 12q + 30$

7. (a) $z^3 + z$

   (b) $15a^2 - 30b$

   (c) $x^2 - x^4$

   (d) $a^3 - 2a^2 + 3a$

   (e) $6x^2 - 144y - 150$

   (f) $3xy + y(x - 3)$

8. (a) $2(z + 1) - 6zw$

   (b) $a^3b^3 + a^2b^4 - a^2b^3$

   (c) $3ab + 4bc - 4ac$

   (d) $abx - aby$

   (e) $(6r^2s)x - (6r^2s)y$

   (f) $(u^2 + v^2)x - (u^2 + v^2)y$

   (g) $x(4x - y) - y(4x - y)$

   (h) $36a^2b^2c^2$

9. What is the degree of each of the following polynomials?

   (a) $3x + 2,\ \ 5 - x,\ \ (3x + 2)(5 - x)$

   (b) $x^2 - 4,\ 2x + 1,\ \ (x^2 - 4)(2x + 1)$

   (c) $2x^3 - 5x^2 + x\ ,\ \ x^2,\ \ x^2(2x^3 - 5x^2 + x)$

   (d) $1\ ,\ \ 7x^5 - 6x + 2,\ 1\cdot(7x^5 - 6x + 2)$

   (e) $x^2 - 3x - 7,\ \ (x^2 - 3x - 7)^2$

   (f) What can you say about the degree of the product of two polynomials if you know the degrees of the polynomials?

12-2. Factoring by the Distributive Property

In many of our applications of the distributive property in previous chapters, we changed indicated products into indicated sums and indicated sums into indicated products. The latter is actually factoring and it gives us an important technique for factoring certain polynomials. We saw some simple instances of this in the preceding section. More complicated examples will be considered here.

Example 1. $4t^2 - 6t^4$. This is a polynomial in t. Using the distributive property we write

$$4t^2 - 6t^4 = 2t^2\,(2 - 3t^2).$$

Example 2. $6a^3b^2 - 3a^2b^3$. This is a polynomial in two variables a and b. The distributive property enables us to write it in various forms, of which some are

$$\begin{array}{rl} \text{(i)} & 3(2a^3b^2 - a^2b^3) \\ \text{(ii)} & ab(6a^2b - 3ab^2) \\ \text{(iii)} & 3a^2(2ab^2 - b^3) \\ \text{(iv)} & 3a^2b^2(2a - b). \end{array}$$

In each case we have factored the polynomial into polynomials. Which factored form is simplest? In forms (i), (ii), (iii) the distributive property can be used to do more factoring. In the case of (iv), the factoring is complete.

Example 3. $5(z - 2) + (z^2 - 2z)$. This is a polynomial in one variable z over the integers. We first write

$$5(z-2) + (z^2 - 2z) = 5(z - 2) + z(z - 2).$$

Then by comparing with the form of the distributive property we have

$$\begin{array}{ccccc} ac & + & bc & = & (a + b)\;c \\ \downarrow & & \downarrow & & \downarrow \\ 5(z - 2) & + & z(z - 2) & = & (5 + z)(z - 2). \end{array}$$

Notice how we identify c with the single number $(z - 2)$.

The distributive property has been used twice, first to obtain $z^2 - 2z = z(z - 2)$ and then to write $5(z - 2) + z(z - 2)$ as the indicated product $(5 + z)(z - 2)$.

We found many uses for the factored form of an integer. Similarly we will find many applications for the factored form of a polynomial.

Example 4. Solve $5(z - 2) + (z^2 - 2z) = 0$.

The result of Example 3 tells us that, for any real number z,

$$5(z - 2) + (z^2 - 2z) = (5 + z)(z - 2).$$

Thus, an equivalent equation is

$$(5 + z)(z - 2) = 0.$$

The truth set of this equation is $\{-5, 2\}$. (Why?) Notice how factorization of polynomials helped us solve the equation.

Problem Set 12-2a

Factor each of the following expressions as far as you can using the distributive property. Which cases illustrate factoring polynomials over the integers?

1. $3x(2xz - yz)$
2. $6s^2t - 3stu$
3. $144x^2 - 216s + 180y$. What have you learned to do with integers that will enable you to find the largest common factor here?
4. $\frac{6}{5} u^2v - \frac{9}{5} uv^2 + 3v^2$
5. $-x^3y^2 + 2x^2y^2 + xy^2$
6. $\frac{1}{6}ab + \frac{5}{18}a^2b - \frac{7}{12}ab^2$
7. $s \sqrt{3} + s^2 \sqrt{6}$
8. $\frac{a^2}{2}\sqrt{\frac{1}{2}} - \frac{2ab}{3\sqrt{2}}$

9. $3ab + 4bc - 5ac$

10. $a(x - 1) + 3(x - 1)$

11. $(x + 3)^2 - 2(x + 3)$

12. $(u + v)x - (u + v)y$

13. $(a - b)a + (a - b)b$

14. $(x + y)(u - v) + (x + y)v$

15. $(r - s)(a + 2) + (s - r)(a + 2)$

16. $3x(x + y) - 5y(x + y) + (x + y)$

17. $6a\sqrt[3]{2} + 15ab\sqrt[3]{2}$

18. $3|x| + 2a|x|$

19. $7y\sqrt{x^2} - 21y^2|x|$

20. $r(u + v) - (u + v)s$

21. $(a + b + c)x - (a + b + c)y$

22. $(a + b + c)(x + y) - (a + b + c)y$

The distributive property has enabled us to factor polynomials such as $x^2 + bx$ and $ax + ab$ into

$$x^2 + bx = x(x + b)$$

and

$$ax + ab = a(x + b),$$

respectively. Suppose now that we consider the polynomial

$$x^2 + bx + ax + ab.$$

You see that we can factor the sum of the first two terms, namely $x^2 + bx$, and the sum of the last two terms, $ax + ab$. Thus

$$\begin{aligned} x^2 + bx + ax + ab &= (x^2 + bx) + (ax + ab) \\ &= x(x + b) + a(x + b). \end{aligned}$$

We have now succeeded in writing our given sum of four terms as a sum of two terms which have a common factor, $(x + b)$. Applying the distributive property for the third time, we obtain

$$\begin{array}{ccccc} rt & + & st & = & (r + s)\quad t \\ \downarrow & & \downarrow & & \downarrow \\ x(x + b) & + & a(x + b) & = & (x + a)(x + b), \end{array}$$

$$x^2 + bx + ax + ab = (x + a)(x + b).$$

Factoring such as we have done here, by grouping terms, depends on the arrangement of the terms. For example, consider the arrangement $x^2 + ab + bx + ax$. This can be written as

$$x^2 + ab + bx + ax = (x^2 + ab) + (b + a)x.$$

In this form, however, there is no common factor in the two terms, so it does not lead to a factorization of the given polynomial. (Why?)

Example 5. Factor $x^2 + 4x + 3x + 12$.

$$\begin{aligned} x^2 + 4x + 3x + 12 &= (x^2 + 4x) + (3x + 12) \\ &= x(x + 4) + 3(x + 4). \end{aligned}$$

$$\begin{array}{ccccc} a\,c & + & bc & = & (a + b)\quad c \\ \downarrow & & \downarrow & & \downarrow \\ x(x + 4) & + & 3(x + 4) & = & (x + 3)(x + 4). \end{array}$$

Thus,

$$x^2 + 4x + 3x + 12 = (x + 3)(x + 4).$$

Again, notice how $(x + 4)$ is treated as a single number when the distributive property is applied the last time.

Example 6. Factor $xz - 8z + x - 8$

$$\begin{aligned} xz - 8z + x - 8 &= (xz - 8z) + (x - 8) \\ &= (x - 8)z + (x - 8)\cdot 1 \\ &= (x - 8)(z + 1) \end{aligned}$$

Let us try another grouping of terms:

$$\begin{aligned} xz - 8z + x - 8 &= xz + x - 8z - 8 \\ &= (xz + x) - (8z + 8) \\ &= x(z + 1) - 8(z + 1) \\ &= (x - 8)(z + 1) \end{aligned}$$

Example 7. Factor $2st + 6 - 3s - 4t$

$$\begin{aligned} 2st + 6 - 3s - 4t &= (2st + 6) - (3s + 4t) \\ &= 2(st + 3) - (3s + 4t) \end{aligned}$$

This grouping leads us nowhere. Perhaps another grouping will be better. Notice that $2st$ and $-3s$ have a common factor s, and also the remaining terms 6 and $-4t$ also have a common factor 2. Therefore, we try

$$\begin{aligned} 2st + 6 - 3s - 4t &= 2st - 3s - 4t + 6 \\ &= s(2t - 3) - 2(2t - 3) \\ &= (s - 2)(2t - 3). \end{aligned}$$

We should not conclude that all polynomials of this kind can be factored by the method of Examples 5, 6, 7. Some polynomials which look like these simply cannot be factored regardless of the grouping. For example, try to factor $2st + 6 - 3s - 2t$.

Problem Set 12-2b

Factor each of the following polynomials. Consider polynomials over the integers whenever possible.

1. $ax + 2a + 3x + 6$
2. $ux + vx + uy + vy$
3. $2ab + a^2 + 2b + a$
4. $3rs - 3s + 5r - 5$
5. $5x + 3xy - 3y - 5$
6. $3a + 15b - 3a - 15b$

7. $a^2 - ab + ac - bc$

8. $t^2 - 4t + 3t - 12$

9. $p^2 - pq + mp + mq$

10. $2a^2 - 2ab\sqrt{3} - 3ab + 3b^2\sqrt{3}$

11. $15ax + 12bx - 9cx + 6dx$

12. $2a - 2b + ua - ub + va - vb$ (Try three groups of two terms each.)

13. $xu + xv - xw + yu + yv - yw$ (Try two groups of three terms each.)

14. $a^2 - 4ax + 2ab + 3ac - 12cx - 8bx$

15. $\frac{1}{2}axy - a^2y + \frac{1}{6}abx - \frac{1}{3}a^2b$

16. $x^2 + 4x + 3$ (Note that $4x = 3x + x$.)

17. $a^2 - b^2$ (Note that $a^2 - b^2 = a^2 - ab + ab - b^2$.)

12-3. Difference of Squares

Consider, for any two real numbers a and b, the product

$$\begin{aligned}(a + b)(a - b) &= (a + b)a - (a + b)b \\ &= a^2 + ba - ab - b^2 \\ &= a^2 - b^2.\end{aligned}$$

This shows that the product of the sum and difference of any two real numbers is equal to the difference of their squares.

Example 1. Find the product of the sum and difference of 20 and 2.

$$\begin{aligned}(20 + 2)(20 - 2) &= (20)^2 - (2)^2 \\ &= 400 - 4 \\ &= 396.\end{aligned}$$

Example 2. Find the product of the sum and difference of $2x$ and $3y$.

$$\begin{aligned}(2x + 3y)(2x - 3y) &= (2x)^2 - (3y)^2 \\ &= 4x^2 - 9y^2.\end{aligned}$$

Let us turn the above problem around. If we are given the polynomial $a^2 - b^2$, then we know that

$$a^2 - b^2 = (a + b)(a - b).$$

In other words, a difference of squares can be factored into a product of a sum and a difference. Knowing this, we can always factor a polynomial if we can first write it as a difference of squares. Thus, in Example 2, if we are given $4x^2 - 9y^2$, we write

$$\begin{aligned}4x^2 - 9y^2 &= (2x)^2 - (3y)^2 \\ &= (2x + 3y)(2x - 3y).\end{aligned}$$

Example 3. Factor $8y^2 - 18$.

Using the distributive property, we have

$$8y^2 - 18 = 2(4y^2 - 9).$$

In this form we recognize one factor to be the difference of squares:

$$\begin{aligned}4y^2 - 9 &= (2y)^2 - (3)^2 \\ &= (2y + 3)(2y - 3).\end{aligned}$$

Hence,

$$8y^2 - 18 = 2(2y + 3)(2y - 3).$$

Example 4. Factor $3a^2 - 3ab + a^2 - b^2$.

By grouping, we write

$$\begin{aligned}3a^2 - 3ab + a^2 - b^2 &= (3a^2 - 3ab) + (a^2 - b^2) \\ &= 3a(a - b) + (a + b)(a - b) \\ &= (3a + (a + b))(a - b) \\ &= (3a + a + b)(a - b) \\ &= (4a + b)(a - b)\end{aligned}$$

Example 5. Solve the equation $9x^2 - 4 = 0$.

Since $9x^2 - 4 = (3x + 2)(3x - 2)$ for any real number x, the given equation is equivalent to

$$(3x + 2)(3x - 2) = 0.$$

Moreover, for a real number x, $(3x + 2)(3x - 2) = 0$ if and only if either $3x + 2 = 0$ or $3x - 2 = 0$. Therefore the sentence "$9x^2 - 4 = 0$" is equivalent to the sentence "$3x = 2$ or $3x = -2$", and the truth set of the equation $9x^2 - 4 = 0$ is $\{\frac{2}{3}, -\frac{2}{3}\}$.

Problem Set 12-3

1. Do the indicated operations.

(a) $(a - 2)(a + 2)$
(b) $(2x - y)(2x + y)$
(c) $(mn + 1)(mn - 1)$
(d) $(3xy - 2z)(3xy + 2z)$
(e) $(a^2 + b^2)(a^2 - b^2)$
(f) $(x - a)(x - a)$
(g) $(2x - y)(x + 2y)$
(h) $(r^2 - s)(r + s^2)$

2. Factor the following polynomials over the integers if possible.

(a) $4x^2 - 1$
(b) $81 - 9y^2$
(c) $a^2 - 4$
(d) $1 - n^2$
(e) $25x^2 - 9$
(f) $16x^2 - 4y^2$

3. Factor the following polynomials over the integers if possible.

(a) $25a^2 - b^2c^2$
(b) $20s^2 - 5$
(c) $24y^2 - 6z^2$
(d) $16x^3 - 4x$
(e) $16x^2 - 4$
(f) $49x^4 - 1$

4. Factor the following polynomials over the integers if possible.

(a) $x^2 - 4$
(b) $x^2 - 3$
(c) $x^4 - 4$
(d) $x^2 + 4$
(e) $3x^2 - 3$
(f) $16x^4 - 1$

5. Factor the following polynomials over the integers if possible.

(a)	$(a - 1)^2 - 1$	(d)	$(m + n)^2 - (m + n)^2$
(b)	$(a - 1)^2 - (a - 2)^2$	(e)	$(x^2 - y^2) - (x - y)$
(c)	$(m + n)^2 - (m - n)^2$	(f)	$x - y + y^2 - x^2$

6. Solve the equations.

(a)	$x^2 - 9 = 0$	(e)	$4t^3 - t = 0$
(b)	$9r^2 = 1$	(f)	$x^2 + 4 = 0$
(c)	$75s^2 - 3 = 0$	(g)	$y^4 - 16 = 0$
(d)	$2x^2 = 8$	(h)	$(s + 2)^2 - 9 = 0$

7. Factor $20^2 - 1$. Solution:

$$\begin{aligned} 20^2 - 1 &= 20^2 - 1^2 && \text{(Why?)} \\ &= (20 - 1)(20 + 1) && \text{(Why?)} \\ &= 19 \cdot 21 . \end{aligned}$$

Can you see how to reverse these steps? Suppose you are asked to find (19)(21) mentally? Is it easier to find $20^2 - 1$?

Find mentally:

(a)	(22)(18)	(e)	(101)(99)
(b)	(37)(43)	(f)	$(40m)(50n)$
(c)	$(26r)(34)$	(g)	$\big(36(m + n)\big)\big(44(m - n)\big)$
(d)	$(23x)(17y)$	(h)	(6)(6)(4)(11)

8. (a) Can 899 be a prime number? (Hint: $899 = 30^2 - 1$.)

(b) Can 1591 be a prime number?

(c) Can you tell anything about the factors of 391?

(d) Can you tell anything about the factors of 401?

*9. What is $(2 - \sqrt{3})(2 + \sqrt{3})$? Once again, since we have the sum and difference of the same two numbers, this becomes $(2)^2 - (\sqrt{3})^2 = 4 - 3 = 1$. We can apply this to rationalize the denominator in

$$\frac{1}{2 - \sqrt{3}} .$$

By the multiplication property of 1,

$$\frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2+\sqrt{3}}{1} = 2+\sqrt{3}.$$

What does this say about the reciprocal of $2 - \sqrt{3}$?

Of $2 + \sqrt{3}$?

Rationalize the denominator:

(a) $\frac{2}{5+\sqrt{2}}$ (b) $\frac{3+\sqrt{5}}{3-\sqrt{5}}$ (c) $\frac{-6}{2+\sqrt{4}}$ (d) $\frac{\sqrt{3}}{\sqrt{6}-\sqrt{5}}$.

*10. Factor each of the following:

(a) $a^3 + b^3 = a^3 - ab^2 + ab^2 + b^3$

$= a(a^2 - b^2) + (a + b)b^2$

$= a(a + b)(a - b) + (a + b)b^2$

$= (a + b)(a(a - b) + b^2)$

$= (a + b)(a^2 - ab + b^2)$

(b) $t^3 + 1$

(c) $s^3 + 8$

(d) $27x^3 + 1$

*11. Factor each of the following:

(a) $a^3 - b^3 = a^3 - ab^2 + ab^2 - b^3$

$= a(a^2 - b^2) + (a - b)b^2$

$= a(a - b)(a + b) + (a - b)b^2$

$= (a - b)(a(a + b) + b^2)$

$= (a - b)(a^2 + ab + b^2)$

(b) $t^3 - 1$

(c) $s^3 - 8$

(d) $8x^3 - 1$

12-4. Perfect Squares

For any real numbers a and b, consider the product

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= (a + b)a + (a + b)b\\ &= a^2 + ba + ab + b^2\\ &= a^2 + 2(ab) + b^2\end{aligned}$$

The polynomial "$a^2 + 2(ab) + b^2$", since it can be written as the product of two identical factors, is called a perfect square. In the same way we can obtain

$$(a - b)^2 = a^2 - 2(ab) + b^2,$$

so that "$a^2 - 2(ab) + b^2$" is also a perfect square.

The problem is to identify a polynomial which is a perfect square and write it in its "squared" form. We have already met perfect squares of the type $25a^4b^2c^6$ $\left(\text{note that } 25a^4b^2c^6 = (5a^2bc^3)^2\right)$. We are interested here in the two types considered above.

Consider the example $(2x + 3y)^2$ in comparison with the general case $(a + b)^2$.

$$\begin{array}{ccccccccccccc}
(a & + & b)^2 & = & a^2 & + & 2(& a & & b) & + & b^2 \\
\downarrow & & \downarrow & & \downarrow & & & \downarrow & & \downarrow & & \downarrow \\
(2x & + & 3y)^2 & = & (2x)^2 & + & 2(& 2x & \cdot & 3y) & + & (3y)^2
\end{array}$$

$$= 4x^2 + 12xy + 9y^2.$$

If we are given $4x^2 + 12xy + 9y^2$ at the outset, the problem is to write it in the form $(2x)^2 + 2(2x\cdot 3y) + (3y)^2$, from which the factored form $(2x + 3y)^2$ is obtained immediately by taking a as $2x$ and b as $3y$ in the general form.

Example 1. Factor $x^2 + 6x + 9$.

$$\begin{aligned}x^2 + 6x + 9 &= x^2 + 2(3x) + 3^2\\ &= (x + 3)^2\end{aligned}$$

How do you tell at a glance whether or not a polynomial such as this is a perfect square?

Example 2. Factor $9s^2 + 12st + 4t^2$.

$$9s^2 + 12st + 4t^2 = (3s)^2 + 2(3s \cdot 2t) + (2t)^2$$
$$= (3s + 2t)^2.$$

Example 3. Factor $4a^2 - 4ab + b^2$

$$4a^2 - 4ab + b^2 = (2a)^2 - 2(2a \cdot b) + b^2$$
$$= (2a - b)^2.$$

Problem Set 12-4a

1. Fill in the missing term so that the result is a perfect square.

(a) $t^2 - 6t + (\quad)$

(b) $x^2 + 8x + (\quad)$

(c) $a^2 + 12a + (\quad)$

(d) $4s^2 + 4st + (\quad)$

(e) $(\quad) + 6xy + 9y^2$

(f) $u^2 - (\quad) + 25$

(g) $4s^2 + (\quad) + 9$

(h) $9x^2 + 18x + (\quad)$

(i) $4 - 4\sqrt{5} + (\quad)$

(j) $1 + (\quad) + 3$

(k) $5 - (\quad) + 7$

(ℓ) $(\quad) - 8tp + 4p^2$

(m) $(\quad) + 40v + 25$

(n) $49x^2 - (\quad) + 16y^2$

(o) $(v + 1)^2 + 4(v + 1) + (\quad)$

(p) $(x - 1)^2 + (\quad) + 9$

2. Which of the following are perfect squares?

(a) $x^2 + 2xy + y^2$

(b) $x^2 + 2ax + 9a^2$

(c) $7^2 + 2(7)(5) + 5^2$

(d) $7 + 2\sqrt{7}\sqrt{5} + 5$

(e) $4 - 2\sqrt{15} + \frac{15}{4}$

(f) $x^2 - \frac{x}{2} + \frac{1}{4}$

(g) $(x - 1)^2 - 6(x - 1) + 9$

(h) $(2a + 1)^2 + 10(2a + 1) + 25$

3. Factor each of the following polynomials over the integers, if possible.

   (a) $a^2 - 4a + 4$

   (b) $4x^2 - 4x + 1$

   (c) $x^2 - 4$

   (d) $x^2 + 4$

   (e) $4t^2 + 12t + 9$

   (f) $7x^2 + 14x + 7$

   (g) $y^2 + y + 1$

   (h) $4z^2 - 20z + 25$

   (i) $9s^2 + 6st + 4t^2$

   (j) $9(a - 1)^2 - 1$

   (k) $20 - 5x$

   (ℓ) $2a^2 - 20ab + 5b^2$

   (m) $rs + 2st + 3rt$

   (n) $u(v + w) + w(v - w)$

   (o) $2a^3 - 20a^2b + 50ab^2$

   (p) $(s + 3)^2 + 4(s + 3) + 4$

   (q) $(t^2 - 2t + 1) - (s + 1)^2$

   (r) $x^4 - 2x^2 + 1$

   (s) $z^4 + 16z^2 + 64$

4. Write the result of performing the multiplications:

   (a) $(x + 3)^2 =$

   (b) $(x - 2)^2 =$

   (c) $(x + \sqrt{2})^2 =$

   (d) $(a + b)^2 =$

   (e) $(x - y)^2 =$

   (f) $(x - 1)^2 - a^2 =$

   (g) $\big((x - 1) + a\big)\big((x - 1) - a\big) =$

   (h) $(\sqrt{2} + \sqrt{3})^2 =$

   (i) $(100 + 1)^2 =$

Some polynomials can be factored by combining the methods of perfect squares and differences of squares.

Example 4. Factor $x^2 + 6x + 5$.

We know that $x^2 + 6x + 9$ is a perfect square. This suggests that we should write

$$\begin{aligned} x^2 + 6x + 5 &= x^2 + 6x + 9 - 9 + 5 \quad \text{(to form a perfect square} \\ &= (x^2 + 6x + 9) - 4 \\ &= (x + 3)^2 - 2^2 \quad \text{(to form the difference of squares)} \\ &= (x + 3 + 2)(x + 3 - 2) \\ &= (x + 5)(x + 1) \end{aligned}$$

The method used above, of adding and subtracting a number so as to obtain a perfect square, is called completing the square

Example 5. Solve the equation $x^2 - 8x + 18 = 0$.

Completing the square, we obtain

$$x^2 - 8x + 16 - 16 + 18 = 0$$

$$(x - 4)^2 + 2 = 0$$

We know of no method for factoring this polynomial. Does this guarantee that it cannot be factored? In this case, we can find the truth set without writing the polynomial in factored form.

Since $(x - 4)^2$ is non-negative for all x, $(x - 4)^2 + 2$ is never less than 2. Therefore the truth set of the given equation is empty.

Problem Set 12-4b

1. Factor the following polynomials over the integers using the method of completing the square.

   (a) $x^2 + 4x + 3$ (d) $x^2 - 10x + 24$

   (b) $x^2 - 6x + 8$ (e) $x^2 - 10x - 24$

   (c) $x^2 - 2x - 8$ (f) $(x - 1)^2 - 4(x - 1) - 5$

2. What integer values of p, if any, will make the following polynomials perfect squares?

   (a) $u^2 - 2u + p$ (d) $v^2 - 2pv - 1$

   (b) $x^2 + px + 16$ (e) $x^2 - 8x - p + 20$

   (c) $p^2t^2 + 2pt + 1$

3. Solve

   (a) $y^2 - 10y + 25 = 0$ (e) $(a - 1)^2 - 1 = 0$

   (b) $4t^2 - 20t + 25 = 0$ (f) $z^4 + 16z^2 + 64 = 0$

   (c) $9a^2 + 6a + 4 = 0$ (g) $x^2 - 2x - 8 = 0$

   (d) $a^2 = 4a - 4$ (h) $x^2 - 10x + 24 = 0$

12-5. Quadratic Polynomials

We have already pointed out that factoring can be regarded as a process inverse to simplification. Thus it seems plausible that, if someone were to give us a polynomial which was obtained as a result of simplifying a product, we should be able to reverse the process and discover the original factored form. This can be difficult in general but does work in some special cases, as we have already seen in the preceding sections. In this section, we use this approach to factor quadratic polynomials, that is, polynomials in one variable of degree two.

Let us examine the product

$$\begin{aligned}(x + m)(x + n) &= (x + m)x + (x + m)n \\ &= x^2 + mx + xn + mn \\ &= x^2 + (m + n)x + mn\end{aligned}$$

If we are given a quadratic polynomial such as $x^2 + (m + n)x + mn$, where m and n are specific integers, then it is easy to reverse the process:

$$\begin{aligned}x^2 + (m + n)x + mn &= (x^2 + mx) + (nx + mn) \\ &= (x + m)x + (x + m)n \\ &= (x + m)(x + n).\end{aligned}$$

In fact, this is just another example of factoring by the distributive property discussed in Section 12-2. However, suppose that m and n are replaced by some common names, say 6 and 4. Then

$$(x + 6)(x + 4) = x^2 + 10x + 24.$$

Now we can see how trouble arises. The variables m and n retain their identity and hold the form of the expression, while 6 and 4 become lost in the simplification. The problem in factoring a polynomial such as $x^2 + 10x + 24$ is to "rediscover" the numbers 6 and 4.

Let us look at this example more closely.

$$\begin{aligned}(x + 6)(x \text{ [illegible] } &= x^2 + (6 + 4)x + (6 \cdot 4) \\ &= x^2 + 10x + 24.\end{aligned}$$

Evidently the problem is to write 24 as a product of two factors whose sum is 10. In this case, since the numbers are simple you can probably list in your mind ways of factoring 24,

$$\begin{aligned}&1 \cdot 24 \\ &2 \cdot 12 \\ &3 \cdot 8 \\ &4 \cdot 6\end{aligned}$$

and pick the pair of factors whose sum is 10.

Although this method of procedure is easy when the number of factors is small, it becomes tedious when the number of factors is large; on the other hand, the number of cases which need to be considered can frequently be reduced if we use some of our knowledge about integers.

Example 1. Factor the quadratic polynomial $x^2 + 22x + 72$.

We must find two integers whose product is 72 and whose sum is 22. (Do you recall this problem from Chapter 10?) We have $72 = 2^3 3^2$; so the various factors of 72 appear as products of powers of 2 and 3. Since 22 is even, both integers whose sum is 22 must have a factor 2 and, since 22 is not divisible by 3 one of the integers must involve all of the 3's. This reduces the possibilities to

$$2^2 \cdot 3^2 + 2 = 36 + 2$$

or

$$2^2 + 2 \cdot 3^2 = 4 + 18.$$

Since $4 + 18 = 22$, it follows that

$$\begin{aligned} x^2 + 22x + 72 &= x^2 + (4 + 18)x + 4 \cdot 18 \\ &= (x + 4)(x + 18). \end{aligned}$$

Example 2. Factor $a^2 - 69a - 450$.

The prime factorization of 450 is $2 \cdot 3^2 \cdot 5^2$. Since 69 is not a multiple of 5, the 5's must be together. Since 69 is a multiple of 3, the 3's must be split.

Furthermore, since -450 is negative, one of its factors must be positive and the other negative. Thus, we want to find a positive integer and a negative integer whose sum is -69 and whose product is -450. The possibilities are

$$5^2 \cdot 3 - 3 \cdot 2 = 75 - 6$$

$$5^2 \cdot 3 \cdot 2 - 3 = 150 - 3$$

and their "opposites". The opposite of the first one gives -69; that is, $-(5^2 \cdot 3) + 3 \cdot 2 = -69$. Hence

$$a^2 - 69a - 450 = a^2 + (6 - 75)a + 6(-75)$$
$$= (a + 6)(a - 75).$$

Example 3. Factor $x^2 + 5x + 36$.

The prime factorization of 36 is $2^2 \cdot 3^2$. If we examine all possible pairs of factors of 36, we find that the sum can never be as small as 5.

Product	Sum
$1 \cdot 36$	37
$2 \cdot 18$	20
$3 \cdot 12$	15
$4 \cdot 9$	13
$6 \cdot 6$	12

It appears that the smallest sum occurs when the two factors are equal. Since $5 < 12$, we conclude that $x^2 + 5x + 36$ cannot be factored because the coefficient of x is too small.

By similar reasoning determine whether $x^2 - 10x + 36$ is factorable. Is $x^2 + 13x + 49$ factorable? Is $x^2 + 14x + 49$ factorable?

Next try to factor $x^2 + 40x + 36$. In this case 40 is too large for $x^2 + 40x + 36$ to be factorable. (Notice the preceding table of products and sums.)

By similar reasoning determine whether $x^2 - 38x + 36$ is factorable. Is $x^2 + 51x + 49$ factorable? Is $x^2 - 50x + 49$ factorable?

Problem Set 12-5a

In Problems 1 - 9 factor the quadratic polynomials, if possible, using the above method.

1. (a) $a^2 + 8a + 15$ (c) $a^2 + 2a - 15$

   (b) $a^2 - 8a + 15$ (d) $a^2 - 2a - 15$

2. (a) $t^2 + 12t + 20$
   (b) $t^2 + 21t + 20$
   (c) $t^2 + 9t + 20$
   (d) $t^2 + 10t + 20$

3. (a) $a^2 + 6a - 55$
   (b) $x^2 - 5x + 6$
   (c) $u^2 - 10u + 24$
   (d) $y^2 - 17y - 18$
   (e) $z^2 - 2z + 18$

4. (a) $-x^2 + 7x - 12$
   (b) $12 - 11x - x^2$
   (c) $-x^2 - 4x + 12$
   (d) $-x^2 - 13x - 12$
   (e) $-x^2 + x - 12$

5. (a) $a^2 - 16a + 64$
   (b) $a^2 + 8a + 64$
   (c) $a^2 + 36a + 64$
   (d) $a^2 - 20a + 64$
   (e) $a^2 - 16a - 64$

6. (a) $x^2 - 9$
   (b) $a^2 + 1$
   (c) $d^2 - 2$
   (d) $h^2 - 169$

7. (a) $z^6 - 7z^3 - 8$
   (b) $b^4 - 11b^2 + 28$
   (c) $a^4 - 13a^2 + 36$
   (d) $y^4 - 81$

8. (a) $5a + a^2 - 14$
   (b) $10a + 39 + a^2$
   (c) $108 + a^2 - 21a$
   (d) $a^2 + 25a - 600$

9. (a) $3y^2 - 12y + 12$
   (b) $x^3 + 19x^2 + 34x$
   (c) $5a^3 - 15a^2 + 30a$
   (d) $7x^2 - 63$

10. Solve the equations

(a) $a^2 - 9a - 36 = 0$ (e) $x^2 + 6 = 7x$

(b) $x^2 = 5x - 6$ (f) $(x - 2)(x + 1) = 4$

(c) $y^2 - 13y + 36 = 0$ (g) $6x^2 + 6x - 72 = 0$

(d) $x^2 + 6x = 0$ (h) $x^2 - 4x + 11 = 0$

11. Translate the following into open sentences and find their truth sets:

(a) The square of a number is 7 greater than 6 times the number. What is the number?

(b) The length of a rectangle is 5 inches more than its width. Its area is 84 square inches. Find its width.

(c) The square of a number is 9 less than 10 times the number. What is the number?

12. A rectangular bin is 2 feet deep and its perimeter is 24 feet. If the volume of the bin is 70 cu. ft., what are the length and the width of the bin?

13. Two plywood panels each of which cost 30¢ per square foot, were found to have the same area although one of them was a square and the other was a rectangle 6 inches longer than the square but only 3 inches wide. What were the dimensions of the two panels?

*14. Prove that if p and q are integers and if $x^2 + px + q$ is factorable, then $x^2 - px + q$ is also factorable.

*15. Find every integer p such that $x^2 + px + 36$ is factorable. For which values of p will $x^2 + px + 36$ be a perfect square? How are these values of p distinguished from the other values? Answer the same questions for the polynomial $x^2 + px + 64$. If n is a positive integer, what is your guess as to the smallest positive integer p for which $x^2 + px + n^2$ is factorable? What is your guess as to the largest positive integer p for which $x^2 + px + n^2$ is factorable?

In the quadratic polynomials of Examples 1, 2, 3, the coefficient of the second power of the variable was equal to 1. In order to see how to handle other quadratic polynomials let us again consider a product.

$$\begin{aligned}(ax + b)(cx + d) &= (ax + b)cx + (ax + b)d \\ &= (ac)x^2 + (ad + bc)x + (bd)\end{aligned}$$

In order to simplify our discussion, let us call a and b the coefficients of the factor $(ax + b)$, c and d the coefficients of $(cx + d)$, and ac, $(ad + bc)$, bd the coefficients of the quadratic polynomial $(ac)x^2 + (ad + bc)x + (bd)$.

Notice how the coefficients of the quadratic polynomial arise. The coefficient of x^2 is the product of the first coefficients of the factors, the constant is the product of the constants of the factors, and the coefficient of x is the product of the "outside" coefficients plus the product of the "inside" coefficients. For example:

$$\underset{2\cdot 2 + 5\cdot 3}{\overset{2\cdot 3 \qquad 5\cdot 2}{(2x + 5)(3x + 2)}} = 2\cdot 3x^2 + (2\cdot 2 + 5\cdot 3)x + 5\cdot 2.$$

For simplicity in speaking of these coefficients, we call $2\cdot 3$ the product of the *first* coefficients, $5\cdot 2$ the product of the *last*, $2\cdot 2$ the product of the *outside* and $5\cdot 3$ the product of the *inside* coefficients.

Thus, the problem of factoring such a quadratic polynomial as "$6x^2 + 19x + 10$" is a problem of finding two factors of 6, and two factors of 10 such that the "sum of the products of the outside and inside factors" is 19. In simple cases, this can be done by checking all possible factorizations of the coefficients. Since the factors of 6 are $1\cdot 6$ or $2\cdot 3$ and the factors of 10 are $1\cdot 10$ or $2\cdot 5$ or $5\cdot 2$ or $10\cdot 1$, we try each possibility.

(1) $(1x + 1)(6x + 10)$

$1 \cdot 10 + 1 \cdot 6 = 16$

(2) $(1x + 2)(6x + 5)$

$1 \cdot 5 + 2 \cdot 6 = 17$

(3) $(1x + 5)(6x + 2)$

$1 \cdot 2 + 5 \cdot 6 = 32$

(4) $(1x + 10)(6x + 1)$

$1 \cdot 1 + 10 \cdot 6 = 61$

(5) $(2x + 1)(3x + 10)$

$2 \cdot 10 + 1 \cdot 3 = 23$

(6) $(2x + 2)(3x + 5)$

$2 \cdot 5 + 2 \cdot 3 = 16$

(7) $(2x + 5)(3x + 2)$

$2 \cdot 2 + 5 \cdot 3 = \underline{19}$

Of course, with a little practice we would have gone directly to the desired factors $(2x + 5)$ and $(3x + 2)$ by eliminating the other cases mentally. We would think: $6 = 2 \cdot 3$ and $10 = 2 \cdot 5$. Since the middle coefficient 19 is odd, we cannot have an even factor in each of the outside and inside products. This rules out possibilities (1), (3) and (6). Certainly, the factorization $10 = 1 \cdot 10$ in the other possibilities will give too large a middle coefficient. This leaves us with possibilities (2) and (7). Hence, we try these two and find that (7) is the desired pair of factors.

Next let us consider a quadratic polynomial whose coefficients have many more factors:

$$6x^2 + 7x - 24.$$

We must find a pair of integers whose product is 6, and a pair whose product is -24, such that the sum of the outside and inside products is 7. Again we could check all possible factorizations of 6 and -24, but this time we would have 32 cases. Instead, let us use our knowledge about integers to reduce the number of cases. Since $6 = 2 \cdot 3$ and $24 = 2^3 \cdot 3$, we know that the desired integers will be formed from products of 2's and 3's. Since 7 is odd, we cannot have 2's in both outside and inside products. (Why?) Thus the 2's must all be in the outside product or all in the inside product. Also, since 7 is not divisible by 3, the 3's must be either all in the outside or all in the inside product. Now we have reduced the possibilities to:

$$(1x - 2^3 \cdot 3)(2 \cdot 3x + 1)$$
$$1 \cdot 1 - 2^3 \cdot 3 \cdot 2 \cdot 3 = -143,$$

or

$$(2x - 3)(3x + 2^3)$$
$$2 \cdot 2^3 - 3 \cdot 3 = \underline{7},$$

or their "opposites". Thus, the factored form is found to be

$$6x^2 + 7x - 24 = (2x - 3)(3x + 8).$$

Sometimes it is possible to reduce the number of cases by applying facts about integers; sometimes not. In fact, there is never a guarantee that a given quadratic polynomial can be factored at all.

Example 4. Factor $3x^2 - 2x - 21$.

We look for coefficients such that

$$\underset{-2}{\underbrace{\left(\overset{3}{\overbrace{(\ )x + (\ )}}\right)\left(\overset{-21}{\overbrace{(\ )x + (\ )}}\right)}} = 3x^2 - 2x - 21$$

There is one factorization of 3: $3 \cdot 1$, and two factorizations of 21: $21 \cdot 1$ or $3 \cdot 7$. Since 2 is not divisible by 3, we must keep all the 3's in either the outside or inside product. Of the remaining possibilities, which one yields -2 as the sum of the outside and inside products? Hence,

$$3x^2 - 2x - 21 = (x - 3)(3x + 7)$$

Example 5. Factor $25x^2 - 45x - 36$.

We have $25 = 5^2$ and $36 = 2^2 3^2$. We must find a pair of integers whose product is 25, and a pair of integers whose product is -36, such that the sum of the outside and inside products is -45. Since 5 divides 45, there must be a 5 in each of the outside and inside products. Therefore we must have the first coefficients 5 and 5. Since 3 divides 45, there must also be a 3 in each of the outside and inside products. On the other hand 45 is odd; so the 2's must all occur in one term. We have thus reduced the cases to last coefficients: $12, -3$; or $-12, 3$; or $3, -12$; or $-3, 12$. The case which gives the desired factors is: last coefficients $-12, 3$. Therefore

$$\begin{array}{c} \qquad\qquad\qquad\qquad 25 \qquad -36 \\ 25x^2 - 45x - 36 = (5x - 12)(5x + 3) \\ \qquad\qquad\qquad\qquad -45 \end{array}$$

Problem Set 12-5b

In Problems 1 - 11 factor, if possible, the polynomials over the integers.

1. (a) $2x^2 + 5x + 3$

   (b) $2x^2 + 7x + 3$

   (c) $2x^2 + 9x + 3$

2. (a) $3a^2 + 4a - 7$
   (b) $3a^2 - 4a - 7$
   (c) $-3a^2 - 4a + 7$

3. (a) $4y^2 + 23y - 6$
   (b) $x^2 + 4x - 32$
   (c) $8a^2 + 10a - 3$

4. (a) $3c^2 - 2c - 6$
   (b) $3x^2 - 17x - 6$
   (c) $3y^2 + 3y - 6$

5. (a) $9x^2 - 4$
   (b) $9x^2 - 12x + 4$
   (c) $9x^2 + 12x + 4$

6. (a) $9a^2 + 3a - 2$
   (b) $9a^2 + 3a$
   (c) $9a^2 + 9$

7. (a) $12x^2 - 51x + 45$
   (b) $10x^2 + 43x + 45$
   (c) $10x^2 - 69x - 45$

8. (a) $6 - 23a - 4a^2$
   (b) $6 - 3x^2 + 17x$
   (c) $19x - 6 + 7x^2$

9. (a) $p^2 + 2pq + q^2$

   (b) $4a^2 - 16ab + 7b^2$

   (c) $25x^2 - 70xy + 49y^2$

10. (a) $2a^4 + 20a^3 + 50a^2$

    (b) $a^2b - 9ab + 25b$

    (c) $2a^2 + 15a + 25$

11. Factor:

(a)	$6x^2 - 144x - 150$	(e)	$6x^2 + 25x + 150$
(b)	$6x^2 - 11x - 150$	(f)	$6x^2 + 65x + 150$
(c)	$6x^2 + 60x + 150$	(g)	$6x^2 - 87x + 150$
(d)	$6x^2 - 61x + 150$	(h)	$6x^2 + 63x - 150$

12. Can $2x^2 + ax + b$ be factored if a is even and b is odd? Why?

13. Can $3x^2 + 5x + b$ be factored if 3 is a factor of b? If so, choose a value of b such that $3x^2 + 5x + b$ can be factored.

14. Find the truth set if the domain of the variable is the rational numbers.

    (a) $8x^2 + 10x - 3 = 0$

    (b) $6y^2 + y = 1$

    (c) $6v^2 = 19v + 7$

    (d) $a^2 - 4a + 15 = 0$

15. Find the truth set if the domain of the variable is the rational numbers.

    (a) $9x^2 = 4x$

    (b) $9x^2 = 4$

    (c) $(x - 1)^2 = 4$

    (d) $9(x - 1)^2 = 4$

16. Factor:

(a) $w^2 - 16$ (c) $(y^2 + 6y + 9) - 16$

(b) $(x + 3)^2 - 16$ (d) $a^2 - 10a + 25 - 9b^2$

Translate the following into open sentences and find their truth sets.

17. The sum of two numbers is 15 and the sum of their squares is 137. Find the numbers.

18. The length of a rectangle is 7 inches more than its width and its diagonal is 13 inches. Find its width.

19. One number is 8 less than another, and their product is 84. Find the number.

20. The product of two consecutive odd numbers is 15 more than 4 times the smaller number. What are the numbers?

21. Starting from the same point, Jim walked north at a certain constant rate, while Bill walked west at a constant rate which was 1 m.p.h. greater than that of Jim. If they were 5 miles apart at the end of 1 hour, what was the walking rate of each?

22. The altitude of a triangle is 3 inches less than its base. Its area is 14 square inches. What is the length of its base?

23. Find the dimensions of a rectangle whose perimeter is 28 feet and whose area is 24 square feet.

12-6. Polynomials Over the Rational Numbers or the Real Numbers

Most of the preceding work in factoring was concerned with factoring polynomials over the integers. The name itself suggests that we had in mind the possibility of other kinds of polynomials.

> A phrase formed from rational numbers and variables, with no indicated operations other than addition, subtraction, multiplication and taking opposites, is called a polynomial over the rational numbers.

You give a definition for polynomial over the real numbers.

We thus have three types of polynomials: polynomials over the integers, over the rational numbers, and over the real numbers. Consider the expression $3x^2 - 4x + 1$. This is a polynomial over the integers. Since every integer is also a rational number, $3x^2 - 4x + 1$ may also be thought of as a polynomial over the rational numbers. Is it possible to regard this as a polynomial over the real numbers? The expression $u^3 - \frac{2}{3}u^2 + u - 1$ is a polynomial over the rational numbers. Is it possible to think of it as a polynomial over the integers? Over the real numbers?

The problem of factoring can now be stated more generally:

> The problem is to write a given polynomial,
> which we consider to be of a certain type,
> as an indicated product of polynomials of
> the same type.

Consider the expression "$x^2 - 2$". This is a polynomial over the integers and, as such, can be factored only in the trivial form

$$x^2 - 2 = (-1)(2 - x^2).$$

This is not especially interesting since the factor $2 - x^2$ is not of lower degree than $x^2 - 2$. In this sense, $x^2 - 2$ is "prime" as a polynomial over the integers. However, $x^2 - 2$ can also be considered as a polynomial over the real numbers, and we have

$$\begin{aligned} x^2 - 2 &= x^2 - (\sqrt{2})^2 \\ &= (x + \sqrt{2})(x - \sqrt{2}), \end{aligned}$$

where $x + \sqrt{2}$ and $x - \sqrt{2}$ are polynomials over the real numbers (but not over the rational numbers or integers). Thus $x^2 - 2$, considered as a polynomial over the real numbers, admits a non-trivial factoring. This example shows that it makes a difference in factoring which kind of polynomial is being considered.

Consider the expression

$$"\frac{3}{2}st + \frac{15}{2}st^2 - 27s^2t^3" .$$

This is a polynomial in two variables over the rational numbers. The distributive property enables us to write it in the form

$$(\frac{3}{2})(st + 5st^2 - 18s^2t^3).$$

The factor "$\frac{3}{2}$" may be thought of as a polynomial over the rational numbers, while "$st + 5st^2 - 18s^2t^3$" is a polynomial over the integers. This reduction can always be made:

> A polynomial over the rational numbers can be written as a product of a rational number and a polynomial over the integers.

By this reduction, the problem of factoring polynomials over the rational numbers is reduced to the problem of factoring polynomials over the integers.

Problem Set 12-6

1. Factor each of the following, if possible, considered as
   (i) a polynomial over the rational numbers,
   (ii) a polynomial over the real numbers.

   Example: Considered as a polynomial over the rational numbers,

$$\frac{1}{2}x^2 - 4 = \frac{1}{2}(x^2 - 8).$$

Considered as a polynomial over the real numbers,

$$\frac{1}{2}x^2 - 4 = \frac{1}{2}(x + \sqrt{8})(x - \sqrt{8})$$
$$= \frac{1}{2}(x + 2\sqrt{2})(x - 2\sqrt{2}).$$

(a) $\frac{2}{3}a^2 - \frac{4}{3}$

(b) $17u - 51u^3$

(c) $\frac{1}{2}t^3 - 3t^2 + 4t$

(d) $\frac{1}{2}t^3 - 4t^2 + 8t$

(e) $a^4 - 16$

(f) $4x^2 + 9$

2. Solve the equations

(a) $2x^2 - 6 = 0$

(b) $4t^3 - 8t = 0$

(c) $z^3 + 7z = 0$

3. By checking all possible cases, we see that $x^2 + 4x - 2$, considered as a polynomial over the integers, is not factorable. On the other hand, considered as a polynomial over the real numbers, it may be written

$$x^2 + 4x - 2 = (x^2 + 4x + 4) - 2 - 4,$$

by adding $4 - 4$ to form a perfect square,

$$= (x + 2)^2 - (\sqrt{6})^2$$
$$= (x + 2 + \sqrt{6})(x + 2 - \sqrt{6})$$

Thus, the technique of completing the square so as to obtain the difference of squares sometimes allows us to factor a quadratic polynomial over the real numbers even though it was not factorable as a polynomial over the integers.

Factor the following polynomials over the real numbers, if possible, by completing the square to form differences of squares.

(a) $x^2 + 4x - 1$

(b) $x^2 + 4x + 2$

(c) $x^2 + 4x + 3$

(d) $x^2 - 6x + 6$

(e) $y^2 - 5$

(f) $z^2 - 12z + 34$

(g) $s^2 - 10s + 1$

(h) $2x^2 - 8x - 2$

4. Solve (factor by the method of Problem 3):

(a) $y^2 - 4y + 2 = 0$

(b) $a^2 = 6a - 6$

(c) $t^2 - 10t = 1$

(d) $2v^2 - 4v + 6 = 0$

*5. Observing the pattern of the coefficient of x and the constant in

$$(x + b)^2 = x^2 + 2bx + b^2 ,$$

find the number which makes each of the following a perfect square.

(a) $x^2 + 14x + (\quad)$

(b) $a^2 - 3a + (\quad)$

(c) $y^2 + y + (\quad)$

(d) $m^2 + \frac{2}{5}m + (\quad)$

(e) $b^2 - \frac{3}{4}b + (\quad)$

*6. Factor by completing the square:

(a) $a^2 + 3a + 1$

(b) $y^2 + y - 3$

(c) $x^2 - 5x - 2$

(d) $y^2 + \frac{2}{3}y + \frac{1}{3}$

*7. Solve:

(a) $a^2 + 3a + 1 = 0$

(b) $x^2 = 7x + 3$

(c) $y^2 - \frac{4}{3}y = \frac{5}{9}$

(d) $b^2 + \frac{1}{2}b - \frac{1}{2} = 0$

*8. Factor:

Example: $3x^2 - 12x + 2 = 3(x^2 - 4x + \frac{2}{3})$

$$= 3\left((x^2 - 4x + 4) - 4 + \frac{2}{3}\right)$$

$$= 3\left((x - 2)^2 - \frac{10}{3}\right)$$

$$= 3\left(x - 2 + \sqrt{\frac{10}{3}}\right)\left(x - 2 - \sqrt{\frac{10}{3}}\right)$$

(a) $2x^2 - 12x - 5$

(b) $3y^2 + 2y - 2$

(c) $5a^2 - a - 1$

*9. Solve:

(a) $2x^2 = 12x + 5$

(b) $3a^2 - 6a = 4$

(c) $3m^2 + 5m + 1 = 0$

12-7. The Algebra of Rational Expressions

When we began our discussion of factoring we made a special point of the similarity between factoring polynomials and factoring integers. This similarity can be developed further. The integers are closed under addition, subtraction and multiplication but *not* division. The polynomials are closed under (indicated) addition, subtraction and multiplication but not division. If we extend the system of integers so as also to obtain closure under division (except division by zero) we obtain the system of rational numbers. What is the similar extension for polynomials?

> A *rational expression* is a phrase which involves real numbers and variables with at most the operations of addition, subtraction, multiplication, division and taking opposites.

Is every polynomial a rational expression? Are the rational expressions closed under the indicated operations of addition, subtraction, multiplication and division? Why is $\sqrt{2x - 1}$ not a rational expression?

As examples of rational expressions, we list

(1) $\frac{1}{x} + 1$ (2) $\frac{2x - 3}{4y^2 - 9}$ (3) $\frac{x^3 + 5}{5}$

(4) $\frac{3}{2t} + \frac{5}{s - 1}$ (5) $3a - 2b$ (6) $\frac{z}{z - 1} \cdot \frac{z + 2}{3}$.

Among these rational expressions, (1), (3) and (6) are in one variable. Notice that (2) and (3) are indicated quotients of polynomials, whereas (6) is an indicated product and (1), (4) and (5) are indicated sums of rational expressions. Just as every rational number can be represented as the quotient of two integers, every rational expression can be written as the quotient of two polynomials.

Since rational expressions are phrases, they represent numbers. Therefore, in an expression such as

$$\frac{3}{2t} + \frac{5}{s - 1},$$

the value of 0 is automatically excluded from the domain of the variable t and the value 1 is excluded from the domain of s. Such a restriction is always understood for any phrase which involves a variable in a denominator.

We are now ready to study some of the "algebra" of rational expressions. This amounts to studying the procedures for simplifying (indicated) sums and products of rational expressions to quotients of polynomials. As was the case for the rational numbers, we might expect some of the work we have done with fractions to apply here. In fact, remember that *for each value of its variables a rational expression is a real number.* Therefore, the same properties hold for operations on rational expressions as hold for operations on real numbers.

For real numbers a, b, c, d we have the properties:

(1) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

(2) $\frac{b}{b} = 1$

(3) $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$

(where neither b nor d is zero.)

If we symbolize rational expressions with the capital letters A, B, C, D, we can write corresponding properties:

(i) $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$

(ii) $\frac{B}{B} = 1$

(iii) $\frac{A}{B} + \frac{C}{B} = \frac{A + C}{B}$.

(where neither B nor D can be written as the zero expression.)

What are the restrictions on the domains of the variables involved in B and D? If A, B are rational expressions and B can be written as the zero expression, then is $\frac{A}{B}$ a rational expression?

We use the above properties applied to rational expressions to simplify the rational expressions. In other words, we want to write an expression as a single indicated quotient of two polynomials which do not have common factors.

Example 1. Simplify $\frac{ax - bx}{x^2} \cdot \frac{a^2 + 2ab + b^2}{a^2 - b^2}$.

We use property (i) with $A = ax - bx$, $B = x^2$, $C = a^2 + 2ab + b^2$ and $D = a^2 - b^2$. Each of these polynomials can be factored:

$A = (a - b)x \qquad C = (a + b)(a + b)$

$B = x \cdot x \qquad D = (a + b)(a - b)$.

Hence,

$$\frac{ax - bx}{x^2} \cdot \frac{a^2 + 2ab + b^2}{a^2 - b^2} = \frac{(a - b)x(a + b)(a + b)}{x^2(a + b)(a - b)}, \text{ by (i)}$$

$$= \frac{(a + b)\big(x(a + b)(a - b)\big)}{x\big(x(a + b)(a - b)\big)}$$

$$= \frac{a + b}{x}, \qquad \text{by (ii).}$$

The restrictions are

$$x \neq 0, \quad a \neq b, \quad a \neq -b .$$

[sec. 12-7]

Example 2.

$$\frac{1-x^2}{1+x} \cdot \frac{x-2}{x^2-3x+2} = \frac{(1-x)(1+x)}{1+x} \cdot \frac{x-2}{(x-1)(x-2)}$$

$$= \frac{(1-x)(1+x)(x-2)}{(1+x)(x-1)(x-2)} \qquad \text{(Why?)}$$

$$= \frac{(1-x)\big((1+x)(x-2)\big)}{(x-1)\big((1+x)(x-2)\big)}$$

$$= \frac{1-x}{x-1} \qquad \text{(Why?)}$$

$$= \frac{(-1)(x-1)}{x-1} = -1 \qquad \text{(Why?)}$$

What is the restriction on the domain of x?

Example 3.

$$\frac{\dfrac{x^2+x-2}{x^2-4x+4}}{\dfrac{x+2}{x-2}} = \frac{x^2+x-2}{x^2-4x+4} \cdot \frac{x-2}{x+2}$$

$$= \frac{(x+2)(x-1)(x-2)}{(x-2)(x-2)(x+2)}$$

$$= \frac{x-1}{x-2} \cdot \frac{(x+2)(x-2)}{(x+2)(x-2)}$$

$$= \frac{x-1}{x-2}\ .$$

Problem Set 12-7

Simplify the following, noting restrictions on the values of the variables:

1. $\dfrac{3x-3}{x^2-1}$
2. $\dfrac{x^2-x^2y}{1-y}$
3. $\dfrac{x^2-4x-12}{x^2-5x-6}$
4. $\dfrac{ab+ab^2}{a-ab^2} \cdot \dfrac{1-b}{1+b}$
5. $\dfrac{\dfrac{x^2-9}{6}}{\dfrac{x^2-3x}{3x+3}}$
6. $\dfrac{\dfrac{x^2+2x+1}{x^2-1}}{\dfrac{x+1}{x-1}}$

12-8. Simplification of Sums of Rational Expressions

In order to use the property

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

when adding rational numbers, it is necessary first to write them in a form so that they have a common denominator. In the case of rational numbers, the least (common) denominator which would work was the least common multiple (L.C.M.) of the two given denominators. We have a similar problem in adding rational expressions, and the method is just like that for rational numbers, with factoring of polynomials playing exactly the same role as that of factoring of integers.

Example 1. $\frac{7}{36a^2b} + \frac{5}{24b^3}$

The factored forms of the denominators are

$$36a^2b = 2^2 3^2 a^2 b$$
$$24b^3 = 2^3 3 b^3$$

Choosing each factor the greatest number of times it occurs in either denominator, we find the L.C.M. to be $2^3 \cdot 3^2 a^2 b^3$. Then

$$\frac{7}{36a^2b} + \frac{5}{24b^3} = \frac{7}{2^2 3^2 a^2 b} \cdot \frac{2b^2}{2b^2} + \frac{5}{2^3 3 b^3} \cdot \frac{3a^2}{3a^2}$$

$$= \frac{14b^2}{2^3 3^2 a^2 b^3} + \frac{15a^2}{2^3 3^2 a^2 b^3}$$

$$= \frac{14b^2 + 15a^2}{72a^2b^3}$$

Example 2. $\frac{7}{12 - x - x^2} + \frac{2}{x^2 - 8x + 15}$.

Since $12 - x - x^2 = (4 + x)(3 - x) = (-1)(x + 4)(x - 3)$

and $x^2 - 8x + 15 = (x - 3)(x - 5)$,

the L.C.M. is $(-1)(x - 3)(x + 4)(x - 5)$.

If $x \neq 3$, $x \neq -4$ and $x \neq 5$, then

$$\frac{3}{12-x-x^2} + \frac{2}{x^2-8x+15} = \frac{3}{(-1)(x+4)(x-3)}\cdot\frac{(x-5)}{(x-5)} + \frac{2}{(x-3)(x-5)}\cdot\frac{(-1)(x+4)}{(-1)(x+4)}$$

$$= \frac{3x-15}{(-1)(x+4)(x-3)(x-5)} + \frac{-2x-8}{(-1)(x+4)(x-3)(x-5)}$$

$$= \frac{3x-15-2x-8}{(-1)(x+4)(x-3)(x-5)}$$

$$= \frac{x - 23}{(3-x)(x+4)(x-5)}$$

Example 3. $\frac{a}{3a - 9} - \frac{2a - 3}{5a - 15}$.

The L.C.M. is $3\cdot5(a - 3)$.

If $a \neq 3$, then

$$\frac{a}{3a - 9} - \frac{2a - 3}{5a - 15} = \frac{a}{3(a - 3)}\cdot\frac{5}{5} - \frac{2a - 3}{5(a - 3)}\cdot\frac{3}{3}$$

$$= \frac{5a}{3\cdot5(a - 3)} - \frac{6a - 9}{3\cdot5(a - 3)}$$

$$= \frac{5a - (6a - 9)}{3\cdot5(a - 3)}$$

$$= \frac{5a - 6a + 9}{3\cdot5(a - 3)}$$

$$= \frac{- a + 9}{15(a - 3)}$$

Example 4. $\left(1 - \frac{1}{x+1}\right)\left(1 + \frac{1}{x-1}\right)$

$$1 - \frac{1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1} = \frac{x+1-1}{x+1} = \frac{x}{x+1}$$

and

$$1 + \frac{1}{x-1} = \frac{x-1}{x-1} + \frac{1}{x-1} = \frac{x-1+1}{x-1} = \frac{x}{x-1}.$$

Therefore, if $x \neq 1$ and $x \neq -1$, then

$$\left(1 - \frac{1}{x+1}\right)\left(1 + \frac{1}{x-1}\right) = \frac{x}{x+1} \cdot \frac{x}{x-1} = \frac{x^2}{(x+1)(x-1)} = \frac{x^2}{x^2 - 1}.$$

Problem Set 12-8

1. $\frac{3}{x^2} - \frac{2}{5x}$
2. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$
3. $\frac{1}{a^2} - \frac{1}{2a} - 2$
4. $\frac{5}{x - 1} + 1$
5. $\frac{3}{m - 1} + \frac{2}{m - 2}$
6. $\frac{x}{x + 5} - \frac{x}{x - 3}$
7. $\frac{4}{m - n} + \frac{5}{n}$
8. $\frac{x}{x + y} - \frac{y}{x - y}$
9. $\frac{2}{a - b} - \frac{3}{b - a}$
10. $\frac{5x}{x^2 - 9} + \frac{7}{x + 3}$
11. $\frac{2a}{(a - b)^2} - \frac{3}{a - b}$
12. $\frac{7}{a - b} + \frac{6}{a^2 - 2ab + b^2}$
13. $\frac{3}{x^2 + 2x} - \frac{5}{3x + 6}$
14. $\frac{4}{a^2 - 4a - 5} + \frac{2}{a^2 + a}$
15. $\frac{5}{x^2 + x - 6} + \frac{3}{x^2 - 4x + 4}$
16. $\frac{y - 5}{2y} + \frac{y + 5}{y^2}$
17. $\frac{a}{3 + a} - \frac{a - 3}{a}$
18. $\frac{b + 1}{b - 5} + \frac{b}{10 - 2b}$
19. $\frac{4}{x^2 - x} + \frac{3}{x - 1} - \frac{1}{x}$
20. $\frac{a}{a^2 - 25} - \frac{2}{3a + 15} + \frac{3}{2a - 10}$
21. $\frac{x - \frac{y^2}{x}}{1 + \frac{y}{x}}$ Hint: Multiply by $\frac{x}{x}$.
22. $\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}}$

23. $\dfrac{\frac{x}{3} - 2 + \frac{3}{x}}{1 - \frac{3}{x}}$

24. Consider the set of all rational expressions. Do you think this set is closed under each of the four operations of arithmetic?

12-9. Division of Polynomials.

When you were given a rational number such as $\frac{171}{23}$ in arithmetic you recognized it immediately as an "improper" fraction and hastened to write it in the "proper" form $7\frac{10}{23}$. This form really means $7 + \frac{10}{23}$ and, since $\frac{10}{23} < 1$, does have the advantage of telling immediately that $\frac{171}{23}$ lies between the integers 7 and 8. The number 7 is the integral part of $\frac{171}{23}$. Since

$$7 + \frac{10}{23} = \frac{7 \cdot 23 + 10}{23} = \frac{171}{23},$$

an equivalent way of looking at this is to write

$$171 = 7 \cdot 23 + 10.$$

Thus the integer 171 is represented as an integral multiple of 23 plus an integer which is smaller than 23. This is really what we always do when carrying out the process of dividing one integer by another. How do you check your "answer" in division?

We shall now study the similar problem for rational expressions in one variable. Consider the example:

$$\frac{2x^3 - 6x^2 + 5x + 1}{x^2 - 3x} = \frac{2x(x^2 - 3x) + 5x + 1}{x^2 - 3x}$$

$$= 2x + \frac{5x + 1}{x^2 - 3x}.$$

Does this resemble the above example for rational numbers? Notice that we first wrote the numerator in the form

$$2x^3 - 6x^2 + 5x + 1 = 2x(x^2 - 3x) + (5x + 1),$$

that is, as a polynomial multiple of $x^3 - 3x$ plus a polynomial of lower degree than $x^3 - 3x$. Let us lay aside, for the moment, the question of how we did this (later you will learn a systematic way of doing it) and state in general terms just what the problem is.

> Let N and D be two polynomials in one variable. Then to divide N by D means to obtain polynomials Q and R, with R of lower degree than D, such that
>
> $$\frac{N}{D} = Q + \frac{R}{D} .$$

This problem is equivalent to finding polynomials Q and R such that

$$N = QD + R.$$

As in arithmetic, N is the dividend, D the divisor, Q the quotient, and R the remainder. What are N, D, Q, R in the example considered above? It was easy to obtain Q and R in this example since the first two terms of N, $2x^3 - 6x^2$, contain D as a factor.

Our objective is to give a general step-by-step process for finding polynomials Q and R, being given two polynomials N and D. Notice that, since

$$N = QD + R,$$

it follows that

$$R = N - QD.$$

This means that if we can find Q, then R is obtained simply by subtracting QD from N.

Let us consider another example:

$$\frac{2x^2 + x - 5}{x - 3}$$

Here $N = 2x^2 + x - 5$ and $D = x - 3$. Let us try first to find a polynomial multiple of D which when subtracted from N gives a polynomial of lower degree than N (but not necessarily of lower degree than D). All we need to do is multiply $x - 3$ by a monomial so that the resulting polynomial has the same term of highest degree as N. The highest degree term of N is "$2x^2$". Thus, if we multiply $(x-3)$ by $2x$, the result has the same term of highest degree.

$$2x(x - 3) = 2x^2 - 6x$$

and

$$(2x^2 + x - 5) - 2x(x - 3) = 7x - 5.$$

That is,

$$2x^2 + x - 5 = 2x(x - 3) + (7x - 5). \tag{1}$$

However, the polynomial $7x - 5$ is not of lower degree than $x - 3$. Hence let us apply the same procedure to $7x - 5$. We multiply $(x - 3)$ by 7 in order to have the same highest degree term as $7x - 5$.

$$(7x - 5) - 7(x - 3) = 16$$

$$7x - 5 = 7(x - 3) + 16 \tag{2}$$

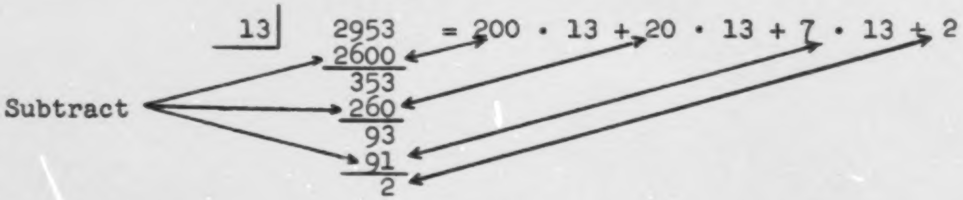
Combining the results (1) and (2), we have

$$\begin{aligned} 2x^2 + x - 5 &= 2x(x - 3) + 7(x - 3) + 16 \\ &= (2x + 7)(x - 3) + 16. \end{aligned}$$

Since 16 has lower degree than $(x - 3)$, the desired polynomials are $Q = 2x + 7$ and $R = 16$. (What is the degree of 16?) Therefore,

$$\frac{2x^2 + x - 5}{x - 3} = (2x + 7) + \frac{16}{x - 3}.$$

In this division process for polynomials we subtract successively (polynomial) multiples of the divisor, obtaining at each step a polynomial of lower degree. We are finished when the result has lower degree than the divisor. This process has probably recalled to you the familiar long division process for numbers in arithmetic -- remember that long division amounts to successive subtractions of multiples of the divisor from the dividend. For example



Thus
$$
\begin{aligned}
2953 &= 200 \cdot 13 + 20 \cdot 13 + 7 \cdot 13 + 2, \\
&= 13(200 + 20 + 7) + 2
\end{aligned}
$$

and
$$
\begin{aligned}
\frac{2953}{13} &= (200 + 20 + 7) + \frac{2}{13} \\
&= 227 + \frac{2}{13} \cdot
\end{aligned}
$$

We can use a similar form for arranging our work in dividing polynomials. First, however, we must see how to arrange subtraction "vertically" as is done in arithmetic. For example, the sentence

$$(-5x^4 + 2x^3 - x + 1) - (3x^4 - x^2 + x + 2) = -8x^4 + 2x^3 + x^2 - 2x - 1$$

may be written "vertically" in the form

$$
\begin{array}{rrrrrr}
 & -5x^4 & + 2x^3 & & - x & + 1 \\
\text{Subtract} \longrightarrow & 3x^4 & & - x^2 & + x & + 2 \\
\hline
 & -8x^4 & + 2x^3 & + x^2 & -2x & - 1.
\end{array}
$$

[sec. 12-9]

Notice that the terms of the same degree are placed one above the other and if a term (such as the second degree term in the first polynomial) is missing, then a space is left for it. The difference is then found by subtracting terms of like degree.

Problem Set 12-9a

1. Subtract, using the "vertical" form described on page 361.

   (a)
$$\begin{array}{rrrr} a^3 & -\ 5a^2 & +\ 2a & +\ 1 \\ \underline{a^3} & \underline{+\ 7a^2} & \underline{+\ 9a} & \underline{-\ 11} \end{array}$$

   (b)
$$\begin{array}{rrrrr} -3x^4 & & 5x^2 & -\ 7x & +\ 2 \\ \underline{-3x^4} & \underline{+\ 2x^3} & \underline{-\ 3x^2} & \underline{} & \underline{-\ 6} \end{array}$$

   (c)
$$\begin{array}{rrr} 7y^2 & +\ 8y & -\ 5 \\ \underline{9y^2} & \underline{+\ 4y} & \underline{} \end{array}$$

2. Use the "vertical" form in the following:

   (a) Subtract $3a^2 - 6a + 9$ from $3a^2 + 7a - 11$.

   (b) From $12x^3 - 11x^2 + 3$ subtract $12x^3 + 6x + 9$.

   (c) Add $14y^2 + 8y - 16$ and $-12y^2 + 3y$.

   (d) From $-6x + 8$ subtract $-6x - 1$.

We can now set up a "vertical" form for dividing polynomials. Let us use the same example as before.

Example 1. $\dfrac{2x^2 + x - 5}{x - 3}$.

$$\begin{array}{lrl} & \underline{x - 3} \big| & 2x^2 + x - 5 = 2x(x - 3) + 7(x - 3) + 16 \\ & & \underline{2x^2 - 6x} \\ \text{Subtract} & & \quad\quad 7x - 5 \\ & & \quad\quad \underline{7x - 21} \\ & & \quad\quad\quad 16 \end{array}$$

Thus, $2x^2 + x - 5 = (2x + 7)(x - 3) + 16$,

and $\dfrac{2x^2 + x - 5}{x - 3} = 2x + 7 + \dfrac{16}{x - 3}$

Problem Set 12-9b

Perform the indicated divisions, using the form shown in Example 1.

1. $\dfrac{2x^2 - 4x + 3}{x - 2}$

2. $\dfrac{4x^2 - 4x - 15}{2x + 3}$

3. $\dfrac{2x^3 - 5x^2 - 8x + 10}{2x + 3}$

4. $\dfrac{2x^3 - 2x^2 + 5}{x - 6}$ Hint: Write the dividend $2x^3 - 2x^2 \qquad + 5$ to allow for the missing first degree term.

5. $\dfrac{2x^5 + x^3 - 5x^2 + 2}{x - 1}$

6. $\dfrac{3x^3 - 2x^2 + 14x + 5}{3x + 1}$

Example 1. on page number 362 can be written more compactly as

$$\begin{array}{lrl}
\text{Dividend} \longrightarrow & & \\
\text{Divisor} \longrightarrow \underline{x - 3}\,\big| & 2x^2 + x - 5 & \big|\underline{2x + 7} \longleftarrow \text{Quotient} \\
& \underline{2x^2 - 6x\phantom{{}-5}} & \\
& 7x - 5 & \\
& \underline{7x - 21} & \\
& 16 & \longleftarrow \text{Remainder}
\end{array}$$

Check: $(2x + 7)(x - 3) + 16 = 2x^2 + x - 5.$

Therefore, $\dfrac{2x^2 + x - 5}{x - 3} = 2x + 7 + \dfrac{16}{x - 3}.$

Example 2. Divide $x^3 + 3x^2 - 38x - 10$ by $x - 5$.

$$
\begin{array}{r|rrrr|l}
\underline{x - 5} & x^3 & +\,3x^2 & -\,38x & -\,10 & \underline{x^2 + 8x + 2} \\
 & x^3 & -\,5x^2 & & & \\ \cline{2-3}
 & & 8x^2 & -\,38x & -\,10 & \\
 & & 8x^2 & -\,40x & & \\ \cline{3-5}
 & & & 2x & -\,10 & \\
 & & & 2x & -\,10 & \\ \cline{4-5}
 & & & & 0 &
\end{array}
$$

Check: $(x^2 + 8x + 2)(x - 5) + 0 = x^3 + 3x^2 - 38x - 10$

Therefore, $\dfrac{x^3 + 3x^2 - 38x - 10}{x - 5} = x^2 + 8x + 2$

Example 3. Divide $3x^3 + x$ by $x + 2$.

$$
\begin{array}{r|rrrr|l}
\underline{x + 2} & 3x^3 & & +\,x & & \underline{3x^2 - 6x + 13} \\
 & 3x^3 & +\,6x^2 & & & \\ \cline{2-3}
 & & -\,6x^2 & +\,x & & \\
 & & -\,6x^2 & -\,12x & & \\ \cline{3-4}
 & & & 13x & & \\
 & & & 13x & +\,26 & \\ \cline{4-5}
 & & & & -\,26 &
\end{array}
$$

Check: $(3x^2 - 6x + 13)(x + 2) - 26 = 3x^3 + x$.

Therefore, $\dfrac{3x^3 + x}{x + 2} = 3x^2 - 6x + 13 - \dfrac{26}{x + 2}$.

Problem Set 12-9c

1. Divide $x^3 - 3x^2 + 7x - 1$ by $x - 3$.
2. Divide $x^2 - 2x + 15$ by $x - 5$.
3. Divide $x^4 - 9x^2 - 1$ by $x + 3$.
4. Divide $5x^3 - 11x + 7$ by $x + 2$.

In Problems 5 - 11 perform the indicated division.

5. $\frac{4x^2 - 4x - 15}{2x + 3}$

6. $\frac{6x^3 - x^2 - 5x + 4}{3x - 2}$

7. $\frac{x^4 - 1}{x - 1}$

8. $\frac{x^4 + 1}{x + 1}$

9. $\frac{x^5 + 1}{x + 1}$

10. $\frac{x^2 - 3x - 9}{3x - 15}$

11. $\frac{x^3 + x - 1}{2x + 1}$

12. $\frac{3x^2 + 2x}{2x + 1}$

13. How will the division process tell you when the polynomial D is a factor of N? Show that $x + 3$ is a factor of

$$2x^4 + 2x^3 - 7x^2 + 14x - 3 .$$

---

*In the above examples and problems, we have emphasized the case in which the divisor is a polynomial of degree one. However, the process works equally well with any two polynomials.

Example 4. $\frac{4x^3 - x^2 + 1}{x^4 - 2x^3 + 1}$

There is nothing to do here since N is already of lower degree than D; so Q = 0 and N = R. In other words, the rational expression is already "proper".

Example 5.

$$\begin{array}{r|l}
x^3 + 2 & 5x^4 - 3x^3 + 2x^2 + 12x + 1 = 5x(x^3 + 2) - 3(x^3 + 2) + 2x^2 + 2x + 7 \\
 & \underline{5x^4 \qquad\qquad\qquad + 10x} \\
 & \qquad - 3x^3 + 2x^2 + 2x + 1 \\
 & \qquad \underline{- 3x^3 \qquad\qquad\quad - 6} \\
 & \qquad\qquad\quad 2x^2 + 2x + 7
\end{array}$$

Thus

$$5x^4 - 3x^3 + 2x^2 + 12x + 1 = (5x - 3)(x^3 + 2) + 2x^2 + 2x + 7$$

and

$$\frac{5x^4 - 3x^3 + 2x^2 + 12x + 1}{x^3 + 2} = 5x - 3 + \frac{2x^2 + 2x + 7}{x^3 + 2} .$$

Written in the more compact form, this becomes:

$$\begin{array}{r|rrrrr|l}
\underline{x^3 + 2} & 5x^4 & - 3x^3 & + 2x^2 & + 12x & + 1 & \underline{5x - 3} \\
 & 5x^4 & & & + 10x & & \\ \cline{2-6}
 & & - 3x^3 & + 2x^2 & + 2x & + 1 & \\
 & & - 3x^3 & & & - 6 & \\ \cline{3-6}
 & & & 2x^2 & + 2x & + 7 &
\end{array}$$

Check: $(5x - 3)(x^3 + 2) + (2x^2 + 2x + 7) = 5x^4 - 3x^3 + 2x^2 + 12x + 1.$

Therefore,

$$\frac{5x^4 - 3x^3 + 2x^2 + 12x + 1}{x^3 + 2} = 5x - 3 + \frac{2x^2 + 2x + 7}{x^3 + 2}.$$

Problem Set 12-9d

1. Perform the indicated divisions.

(a) $\dfrac{2x^2 - 4x + 3}{x - 3}$

(b) $\dfrac{4x^2 - 4x - 13}{2x + 3}$

(c) $\dfrac{2x^3 - 5x^2 - 8x + 9}{2x + 3}$

(d) $\dfrac{3x^3 + 7x^2 - 5x + 4}{3x - 1}$

(e) $\dfrac{x^3 + x - 2}{x + 2}$

(f) $\dfrac{2x^3 - x^2 + 4x + 5}{x^2 - 3}$

(g) $\dfrac{x^3 - 2x^2 + 7x - 1}{x^2 - 2x - 1}$

(h) $\dfrac{3x^4 + 4x^3 - 2x^2 - 7}{x^2 - 4}$

(i) $\dfrac{5x^4 - 3x^3 - x + 4}{5x^3 - 2x^2}$

(j) $\dfrac{3x^4 - 2x^2 - x + 2}{x^3 - x}$

2. Obtain the second factor in each of the following:

   (a) $9x^6 - 25x^4 + 3x + 5 = (3x + 5)(\qquad)$

   (b) $x^9 + 1 = (x^3 + 1)(\qquad)$

   (c) $2x^4 - 5x^2 - x + 1 = (x^2 - x - 1)(\qquad)$

   (d) $4x^8 + 2x^5 - 20x^4 + x^2 - 10x + 25 = (2x^4 + x - 5)(\qquad)$

12-10. Summary

We introduced the concept of a polynomial and saw that the problem of factoring expressions is significant only when restricted to factoring polynomials. Although most of our work was with polynomials over the integers, we also considered polynomials over the rational numbers and over the real numbers. Each of these sets of polynomials is closed under addition and multiplication.

In factoring a polynomial of a given type, we insist on factors which are polynomials of the same type. A polynomial which is not factorable as a polynomial over the integers may or may not be factorable when regarded as a polynomial over the real numbers. Factoring a polynomial over the rational numbers can be reduced to factoring a polynomial over the integers.

We found that factoring is a useful tool for solving equations.

The various methods of factoring polynomials are based on the following forms:

Distributive Property:

$$ab + ac = a(b + c).$$

Difference of Squares:

$$a^2 - b^2 = (a + b)(a - b).$$

Perfect Squares:

$$a^2 + 2(ab) + b^2 = (a + b)^2$$

$$a^2 - 2(ab) + b^2 = (a - b)^2$$

Completing the Square:

$$x^2 + px = (x + \frac{p}{2})^2 - \frac{p^2}{4}.$$

Quadratic Polynomials in one variable:

$$x^2 + (m + n)x + (mn) = (x + m)(x + n).$$

$$(ac)x^2 + (ad + bc)x + (bd) = (ax + b)(cx + d).$$

We considered the concept of a rational expression and observed that rational expressions have the same relationship to polynomials as rational numbers have to integers. Problems of simplifying rational expressions are similar to the similar problems for rational numbers. We saw that rational expressions have the usual properties of fractions and that factoring of polynomials plays the same role in the work with rational expressions as factoring of integers plays in the work with rational numbers.

Every rational expression can be written as an indicated quotient of two polynomials which do not have common factors.

We developed a systematic method for division of polynomials in one variable. This is based on the following important property of polynomials:

> For any two polynomials N and D with D different from zero, there exist polynomials Q and R, with R of lower degree than D, such that $N = QD + R$.

The division process gives us a way of calculating Q and R when N and D are given.

Review Problems

1. Which of the following are rational expressions? polynomials? Which are polynomials in one variable? polynomials over the integers? over the rational numbers? over the real numbers?

   (a) $(s^2 - t)(3st + 1) + 5(s + t)$

   (b) $7x^2 + 2x - 5$

   (c) $ax^2 + bx + c$

   (d) $2u + v$

   (e) $2(u + \frac{1}{2}v)$

   (f) $u + \frac{1}{2}v$

   (g) $\frac{1}{2}(4u + 2v)$

   (h) $(\frac{1}{x})^2 - 2(\frac{1}{x}) + 1$

   (i) $\sqrt{x}$

   (j) $\sqrt[3]{x}$

   (k) $(|x| + 1)(|x| - 1)$

   (ℓ) $\frac{a - b}{a + b}$

   (m) $\frac{a^2 - b^2}{a + b}$

   (n) $\frac{r - 5}{s + 2} \cdot \frac{r + 5}{s - 2}$

   (o) $\frac{r - 5}{s + 2} \cdot \frac{s + 2}{r - 5}$

   (p) $\frac{1}{3 - z} + \frac{z}{2 + z}$

   (q) $s^2 - 2$

   (r) $(s + \sqrt{2})(s - \sqrt{2})$

(s) $(a + b)(a - b) + b^2 - a^2$

(t) $\dfrac{1}{(a + b)(a - b) + b^2 - a^2}$

(u) $(t^2 + 1) \cdot \dfrac{r^3 - 3s^2}{t^2 + 1}$

(v) $\sqrt{4}\, x^2 - \frac{10}{2}x + 1$

(w) $x^2 - 2\sqrt{3}\, x + 3$

2. Simplify the given expressions.

(a) $2\sqrt{18} + 3\sqrt{12} - \sqrt{\frac{1}{2}} - 6\sqrt{\frac{1}{3}}$

(b) $\sqrt{3}\,\sqrt{6a^4}$

(c) $\sqrt{(x + y)^3}$

3. Multiply the factors and simplify the resulting expression.

(a) $2\sqrt{3}\,(2 - \sqrt{6})$

(b) $(\sqrt{3} + \sqrt{2})^2$

(c) $(\sqrt{x} + 1)(\sqrt{x} - 1)$

4. Factor the given polynomials over the integers, if possible.

(a) $x^2 - 22x - 48$

(b) $x^2 - 22x + 48$

(c) $3a^3b^5 - 6a^2b^3 + 12a^4b^4$

(d) $x^2 - y^2 - 4x - 4y$

(e) $(x^2 - y^2) + 2(x - y)^2 - 3(x - y)^3$

(f) $6a^2 - 19a + 10$

(g) $6a^2 + 11a - 10$

(h) $4(x - y)^3 + 8(x - y)^2 - 2(y - x)^2$

(i) $x^2 + 2ax + a^2 - bx - ba - cx - ca$

5. (a) For what positive integral values of k is the polynomial $x^2 + kx + 12$ factorable over the integers?

   (b) For what positive integral values of k is the polynomial $x^2 + 6x + k$ factorable over the integers?

   (c) Determine the value of k so that $x^2 - 6\sqrt{3}x + k$ is a perfect square.

6. Simplify the following:

   (a) $$\frac{\dfrac{3x^2y^6}{20a^2b^2}}{\dfrac{7(xy^2)^3}{30(ab^2)^2}}$$

   (b) $\dfrac{3}{35a^2} + \dfrac{13}{25ab} - \dfrac{5}{7b^2}$

   (c) $\dfrac{2}{a^2 - ab} + \dfrac{3}{b^2 - ab} + \dfrac{4}{ab}$

   (d) $\dfrac{x}{x^2 - 9} + \dfrac{2x - 5}{x^2 - 4x + 3} - \dfrac{3x}{x^2 + 2x - 3}$

7. Divide the given polynomials and check.

   (a) $\dfrac{x^3 - 4x^2 + x + 6}{x - 3}$

   (b) $\dfrac{3x^4 + 14x^3 - 4x^2 - 11x - 2}{3x + 2}$

   (c) $\dfrac{x^3 - 1}{x + 1}$

   (d) $\dfrac{x^5 - 1}{x - 1}$

8. Find the truth sets for the given sentences.

   (a) $\frac{3}{5x} - \frac{3}{4x} = \frac{1}{10}$

   (b) $\frac{4}{y - 5} = \frac{5}{y}$

(c) $\frac{x + 2}{x + 1} - \frac{x - 1}{x + 2} = 0$

(d) $\frac{5}{n - 3} - \frac{20}{n^2 - 9} = -1$

(e) $\frac{1}{|x - 3|} = 7$

(f) $4x^2 - 243 = x^2$

(g) $3|x|^2 - 2|x| = 0$

(h) $|x|^2 + |x| = 12$

9. For each integer n show that the integer $(n + 3)^2 - n^2$ is divisible by 3.

10. Show whether or not $x - 3$ is a factor of the polynomial $x^4 - 5x^3 + 6x^2 - 3$.

11. The polynomial $5x^{100} + 3x^{17} - 1$ can be written in the form

$$5x^{100} + 3x^{17} - 1 = Q(x^3 - x^2 + 1) + R,$$

where Q and R are polynomials.

(a) What can you say about the degree of R if it is as low as possible.

(b) If R has minimum degree, what is the degree of Q?

12. The polynomial $2x^4 + 1$ can be written in the form

$$2x^4 + 1 = 2(x^3 + x^2 + x + 1)(x - 1) + R$$

where R is an integer.

(a) What is the meaning of this equation? What is its truth set?

(b) If a value of x is given, can you find R without carrying out the division process? Is there a special value of x for which this is easiest?

13. The polynomial $5x^{100} + 3x^{17} - 1$ can be written in the form

$$5x^{100} + 3x^{17} - 1 = Q(x - 1) + R,$$

where Q is a polynomial and R is an integer. Find the integer R without carrying out the division process.

14. The polynomial $4x^8 + n$, where n is an integer, can be written in the form

$$4x^8 + n = Q(x - 1) + R,$$

where R is an integer.

(a) What is the degree of Q?

(b) If $R = 0$, what can you say of the relationship between $(x - 1)$ and $4x^8 + n$?

(c) Choose an integer value for n so that $R = 0$.

15. Given the polynomials $2x^{17} - 5x^{15} + 1$ and $x + 3$, find polynomials Q and R such that

$$2x^{17} - 5x^{15} + 1 = Q(x + 3) + R$$

where R has degree less than 16.

16. Prove the

Theorem: If a and b are distinct positive real numbers,

then $$\frac{a + b}{2} > \sqrt{ab} .$$

Hint: Observe that proving $\frac{a + b}{2} > \sqrt{ab}$ is equivalent to proving $a + b - 2\sqrt{ab} > 0$.

In problems 17 through 29, translate the problem into an equation or inequality, and solve the problem by finding the truth set of the equation or inequality.

17. One boy can do his paper route in 30 minutes. His substitute did the route in reverse order and required 45 minutes. Working together, toward each other, how long would it be before they met?

18. A candy store made a 40 lb. mixture of creams selling at \$1.00 per pound and nut centers selling at \$1.40 per pound. If the mixture is to sell at \$1.10 per pound, how many pounds of each kind of candy should be used?

19. A 100 gallon container is tested and found to contain 15% salt. How much of the 100 gallons should be withdrawn and replaced by pure water to make a 10% solution?

20. A jet travels 10 times as fast as a passenger train. In one hour the jet will travel 120 miles further than the passenger train will go in 8 hours. What is the rate of the jet? the train?

21. Two trains 160 miles apart travel towards each other. One is traveling $\frac{2}{3}$ as fast as the other. What is the rate of each if they meet in 3 hours and 12 minutes?

22. A man makes a trip of 300 miles at an average speed of 30 miles per hour and returns at an average speed of 20 miles per hour. What was his average speed for the entire trip?

23. Generalizing problem 22: a man makes a trip of d miles at an average speed of r miles per hour and returns at an average rate of q miles per hour; what was his average rate for the entire trip?

24. The sum of the reciprocals of two successive integers is $\frac{27}{182}$. What are the numbers?

25. Find the average of $\frac{x + 3}{x}$ and $\frac{x - 3}{x}$.

26. The square of a number is 91 more than 6 times the number. Write a corresponding equation and find its truth set.

27. One automobile travels a distance of 360 miles in 1 hour less than a second going 4 miles per hour slower than the first. Find the rate of the two automobiles.

28. A rug with area of 24 square yards is placed in a room 14 feet by 20 feet leaving a uniform width around the rug. How wide is the strip around the rug? A sketched diagram of the rug upon the floor may help you represent algebraically the length and width of the rug.

29. One leg of a right triangle is 2 feet more than twice the smaller leg. The hypotenuse is 13 feet. What are the lengths of the legs?

30. Tell which of these numbers are rational:

$$\sqrt[3]{\pi^3}\ ,\quad \sqrt{.4}\ ,\quad \sqrt[3]{.0008}\ ,\quad (\sqrt[3]{-1})(\sqrt{.16})\ .$$

31. If a two-digit number of the form $10t + u$ is divided by the sum of its digits, the quotient is 4 and the remainder is 3. Find the numbers for which this is true.

32. Simplify $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}$

33. Find the truth set of $|x - 5|^2 \geq 9$.

34. At what time between 3 and 4 o'clock will the hands of a clock be together? At what time will they be opposite each other?

35. A farmer has $1000 to buy steers at $25 and cows at $26. If you know the number of steers and the number of cows are each positive integers, what is the greatest number of animals he may buy, if he must use the entire $1000.

Chapter 13

TRUTH SETS OF OPEN SENTENCES

13-1. Equivalent Open Sentences

Throughout this course we have been solving open sentences, that is, finding their truth sets. At first we guessed values of the variable which made the sentence true, always checking to verify the truth of the sentence. Later we learned that certain operations, when applied to the members of a sentence, yielded other sentences with exactly the same truth set as the original sentence. We say that:

> Two sentences are equivalent if they have the same truth set.

Our procedure for solving a sentence then consisted of performing permissible operations on the sentence to yield an equivalent sentence whose truth set is obvious.

What are such permissible operations? Let us recall a problem from our previous work.

Example 1. Solve $3x + 7 = x + 15$.

This sentence is equivalent to

$$(3x + 7) + (-x - 7) = (x + 15) + (-x - 7)$$

that is, to

$$2x = 8.$$

This sentence is equivalent to

$$(2x)(\tfrac{1}{2}) = (8)(\tfrac{1}{2}),$$

that is, to

$$x = 4.$$

Hence, "$3x + 7 = x + 15$" and "$x = 4$" are equivalent sentences, and the desired truth set is $\{4\}$.

Let us examine this example closely. When we say that "$3x + 7 = x + 15$" is equivalent to "$2x = 8$", we mean that every solution of the first sentence is a solution of the second and every solution of the second sentence is a solution of the first. How can we be sure of this? We know that $(-x - 7)$ is a real number for every value of x. Thus, when we add $(-x - 7)$ to both members of the first sentence we obtain another sentence which is true for the same values of x and possibly more. To show equivalence of these sentences we must also verify that every solution of this second sentence is a solution of the first. This we could do by adding $(x + 7)$ to the members of the second sentence to obtain the first sentence, thus showing that every solution of the second sentence is a solution of the first.

The point is that we did not need to perform this second step of "reversing" the operation. We knew it was possible, because we know that the opposite of $(-x - 7)$ is also a real number for every value of x.

In the same way we know that "$2x = 8$" is equivalent to "$x = 4$", because the operation of multiplying the members of "$2x = 8$" by $\frac{1}{2}$ has an inverse operation, that of multiplying by 2. In fact, every non-zero real number has a reciprocal which is a real number.

Thus, two operations which yield equivalent sentences are:

(1) adding a real number to both members,

(2) multiplying both members by a non-zero real number.

All the sentences we solved thus far have been of a type whose members are polynomials. Recall that a polynomial involves no indicated division with variables in the denominator. As a result we have not needed to face the problem of finding simpler sentences by multiplying members of a sentence by expressions involving variables.

Let us now consider other types of sentences, including those whose members are rational expressions.

Example 2. Solve $\frac{x^2}{x^2 + 1} = \frac{1}{2}$.

By multiplying both members by $2(x^2 + 1)$ we can obtain a sentence free of fractions. Will this operation yield an equivalent sentence? Yes, because for every value of x, $2(x^2 + 1)$ is a non-zero real number. Thus, the sentence is equivalent to

$$\frac{x^2}{x^2 + 1} \cdot 2(x^2 + 1) = \frac{1}{2} \cdot 2(x^2 + 1),$$

that is, to

$$2x^2 = x^2 + 1.$$

This sentence is equivalent to

$$x^2 - 1 = 0, \qquad \text{(Why?)}$$

that is, to

$$(x - 1)(x + 1) = 0.$$

Finally, this sentence is equivalent to

$$x - 1 = 0 \quad \text{or} \quad x + 1 = 0, \qquad \text{(Why?)}$$

and we find the desired truth set to be $\{1, -1\}$.

Problem Set 13-1a

1. For each of the following pairs of sentences, determine whether or not the sentences are equivalent. You can prove they are equivalent by beginning with either sentence and applying operations that yield equivalent sentences, until you arrive at the other sentence of the pair. If you think they are not equivalent, try to prove it by finding a number that is in the truth set of one, but not in the truth set of the other.

   (a) $2s = 12$; $s = 6$

   (b) $5s = 3s + 12$; $2s = 12$

   (c) $5y - 4 = 3y + 8$; $y = 6$

   (d) $7s - 5s = 12$; $s = 6$

(e) $2x^2 + 4 = 10$; $x^2 = 4$

(f) $3x + 9 - 2x = 7x - 12$; $\frac{7}{3} = x$

(g) $x^2 = x - 1$; $1 = x - x^2$

(h) $\frac{y - 1}{|y| + 2} = 3$; $y - 1 = 3(|y| + 2)$

(Hint: Is $(|y| + 2)$ a non-zero real number for every value of y ?)

(i) $x^2 + 1 = 2x$; $(x - 1)^2 = 0$

(j) $x^2 - 1 = x - 1$; $x + 1 = 1$

(k) $\frac{x^2 + 5}{x^2 + 5} = 0$; $x^2 + 5 = 0$

(ℓ) $\frac{x^2 + 5}{x^2 + 5} = 1$; $x^2 + 5 = 1$

(m) $v^2 - 1 = 0$; $|v + 1| = 0$

2. Decide in each pair of sentences whether they are equivalent.

(a) $4 - 2x = 10$; $x = -3$

(b) $12x + 5 = 10 - 3x$; $x = \frac{1}{3}$

(c) $x^2 - 4 = 0$; $x = 2$ or $x = -2$

(d) $x = 3$; $x(x - 3) = 0$

(e) $x - 1 = 0$; $x^2 - 1 = 0$

(f) $|x| = 1$; $x^2 = 1$

3. Change each of the following to a simpler equivalent equation.

(a) $y + 23 = 35$

(b) $\frac{19}{20}x = 19$

(c) $6 - t = 7$

(d) $\frac{1}{7}s = \frac{1}{105}$

(e) $x(x^2 + 1) = 2x^2 + 2$

(Hint: Is $\frac{1}{x^2 + 1}$ a non-zero real number for every value of x ?)

(f) $y(|y| + 1) = |y| + 1$

4. Solve (that is, find the truth set of), if possible:

(a) $11t + 21 = 32$

(b) $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$

(c) $\frac{5}{8}x - 17 = 33$

(d) $6 - s = s + 6$

(e) $s - 6 = 6 - s$

(f) $s - 6 = s + 6$

(g) $\frac{y}{3} + \frac{2}{3} = \frac{y}{2} + \frac{3}{2}$

(h) $4x + \frac{3}{2} = x + 6$

(i) $x^4 + x^2 + 1 = x^2$

(j) $y^4+y^3+ y^2+y+1 = y^4-y^3+y^2-y+1$

(k) $x^2 + 3x = x + \frac{x^2}{2}$

5. Often we can simplify one or both members of a sentence. What kinds of algebraic simplification will guarantee that the simplified form is equivalent to the original? Consider combining terms:

Are $3x - 2 - 4x + 6 = 0$ and $-x + 4 = 0$ equivalent?

Consider factoring:

Are $x^2 - 5x + 6 = 0$ and $(x - 3)(x - 2) = 0$ equivalent?

Are $\frac{x^2 - 4}{x - 2} = 4$ and $x + 2 = 4$ equivalent?

6. In each of the following pairs of sentences, tell why they are equivalent or why they are not equivalent.

(a) $2a + 5 - a = 17$; $a + 5 = 17$

(b) $3x^2 - 6x = 0$; $3x(x - 2) = 0$

(c) $3x^2 = 6x$; $3x = 6$

(d) $3x^2 = 6x$; $3x^2 - 6x = 0$

(e) $6y^2 + 3 - 2y^2 = 5 + y + 2$; $4y^2 + 3 = y + 7$

(f) $b + 3 = 0$; $0 = b + 3$

(g) $2 = \frac{y^2 + 2y}{y}$; $2 = y + 2$

(h) $2(h + 2) + 2(h + 3) = 27$; $4h + 10 = 27$

We have been careful to add only real numbers or multiply only by non-zero real numbers, because we are sure that such operations yield equivalent sentences. Is it possible that other operations may also yield equivalent sentences? Let us look at another example.

Example 3. Solve $x(x - 3) = 2(x - 3)$.

Without any formal operations we can guess that 2 and 3 are solutions of this equation. Are there others? In an attempt to find a simpler equivalent sentence, we might be tempted to multiply both members by $\frac{1}{x - 3}$. Then we obtain the new sentence

$$x(x - 3)\cdot\frac{1}{x - 3} = 2(x - 3)\cdot\frac{1}{x - 3},$$

which in simpler form is

$$x = 2.$$

It is certainly true that 2 is the only solution of this sentence. This means that the operation of multiplying by $\frac{1}{x - 3}$ yielded a new sentence with a smaller truth set. Thus, such an operation will not necessarily give an equivalent sentence. You have probably detected the difficulty: for $x = 3$, $\frac{1}{x - 3}$ is not a number, and the operation of multiplying by $\frac{1}{x - 3}$ is sensible only for values of x other than 3.

Example 3 suggests that we must never add or multiply both members of a sentence by an expression which for some value of the variable is not a number.

Example 4. Solve $\frac{x - 2}{x - 1} = 2 - \frac{1}{x - 1}$.

We first observe that the domain of x cannot include the number 1. (Why?) Thus, we are really solving the sentence

$$\frac{x - 2}{x - 1} = 2 - \frac{1}{x - 1} \text{ and } x \neq 1.$$

It is natural to multiply both members of the equation by $(x - 1)$. Is $(x - 1)$ a real number for every value of x in its domain? Is $(x - 1)$ non-zero? (Remember that $x \neq 1$.) Therefore, we obtain an equivalent sentence by multiplying by $(x - 1)$:

$$\frac{x - 2}{x - 1}\cdot(x - 1) = 2(x - 1) - \frac{1}{x - 1}\cdot(x - 1) \text{ and } x \neq 1,$$

$$x - 2 = 2x - 2 - 1 \text{ and } x \neq 1,$$

$$1 = x \text{ and } x \neq 1.$$

This latter sentence has an empty truth set. Hence, the original sentence has no solutions.

The problem in Example 4 points out that we must be careful to keep a record of the domain of the variable. Thus, we may multiply by an expression which for all values in the domain of the variable is a non-zero real number.

Problem Set 13-1b

1. For each of the following phrases decide whether it is
   (i) a real number for every value of the variable,
   (ii) a non-zero real number for every value of the variable.

(a) $x^2 - 4x + 3$

(b) $\frac{3 - 4y}{y + 4}$

(c) $3 + r + \frac{1}{r}$

(d) $\sqrt{t + 1}$

(e) $|y + 1|$

(f) $|y| + 1$

(g) $\frac{x^2}{x^2 + 1}$

(h) $\frac{x^2 + 1}{x^2 + 1}$

(i) $\sqrt{v^2 + 1}$

(j) -3

(k) $\frac{x}{x}$

(ℓ) $\frac{q^2 - 1}{q + 1}$

2. Solve:

(a) $\frac{y}{y - 2} = 3$

(b) $\frac{x}{x^2 + 1} = x$

(c) $\frac{1}{x} + 3 = \frac{2}{x}$

(d) $\frac{1}{x - 2} + \frac{x - 3}{x - 2} = 2$

(e) $-\frac{1}{x + 1} + 1 = \frac{x}{x + 1}$

(f) $x(x^2 + 1) = 2x^2 + 2$

3. Find the dimensions of a rectangle whose perimeter is 30 inches and whose area is 54 square inches.

4. Find three successive integers such that the sum of their squares is 61.

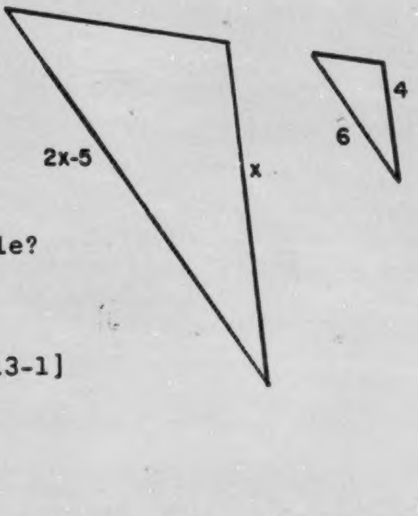
5. The sum of two numbers is 8 and the sum of their reciprocals is $\frac{2}{3}$. What are the numbers?

6. In a certain school the ratio of boys to girls was $\frac{7}{6}$. If there were 2600 students in the school, how many girls were there?

7. Show that any pair of numbers (x, y) for which either one of the equations $3x + 18 = y + 23$, $y = 3x - 5$ is true is also a pair for which the other is true. What, then, is the relation between the set of all solution pairs of the first equation and the set of all solution pairs of the second?

8. Show that the equations $4x - \frac{2}{3}y = 6$ and $y = 6x - 9$ are equivalent.

9. The sides of lengths x and $2x - 5$ of the first triangle shown have the same ratio as the sides of lengths 4 and 6 respectively of the second triangle. How long are the two sides of the first triangle?



[sec. 13-1]

10. A mixture for killing weeds must be made in the ratio of 3 parts of weed-killer to 17 parts of water. How many quarts of weed-killer should be put in a 10 gallon tank which is going to be filled up with water to make 10 gallons of mixture?

13-2. Equivalent Inequalities

In Chapter 8 we solved certain inequalities by obtaining simpler equivalent inequalities. Recall that we often used the properties:

> For real numbers a, b, c, $a < b$ if and only if $a + c < b + c$,

and

> for c positive, $a < b$ if and only if $ac < bc$,
> for c negative, $a < b$ if and only if $ac > bc$.

It turns out that the operations we may perform on an inequality to yield an equivalent inequality are somewhat like those for equations. The only difference is that when we multiply both members of an inequality by a non-zero real number, we must be sure that it is positive or that it is negative. For example, $x^2 + 1$ is always positive for every value of x; $-\frac{1}{x^2 + 2}$ is always negative for every value of x; but $x^2 - 1$ is negative for some values, positive for other values, and 0 for others. Hence, we shall not use $x^2 - 1$ as a multiplier.

To summarize, some operations which yield equivalent inequalities are:

(1) adding a real number to both members,

(2) multiplying both members by a positive number, in which case the order of the resulting products is unchanged,

(3) multiplying both members by a negative number, in which case the order of the resulting products is reversed.

Example 1. Solve $\frac{4}{5}y - 6 < \frac{2}{3}y + \frac{5}{6}$.

We may first multiply both members by the positive real number 30 to obtain a sentence free of fractions:

$$24y - 180 < 20y + 25 .$$

Now we add the real number $-20y + 180$ to both members:

$$4y < 205 .$$

Finally, we multiply by the positive real number $\frac{1}{4}$:

$$y < \frac{205}{4} .$$

What is the truth set of the original inequality? Explain why all these sentences are equivalent.

Example 2. Solve $-\frac{1}{x^2 + 1} > -1$.

Since $-(x^2 + 1)$ is a negative real number for every value of x, we may multiply both members by $-(x^2 + 1)$ to obtain the equivalent sentence

$$1 < x^2 + 1 .$$

By adding -1 to both members, we have the equivalent sentence

$$0 < x^2 .$$

The truth set of this final sentence is the set of all non-zero real numbers. This is also the truth set of the original inequality.

Problem Set 13-2

1. Solve the following inequalities by changing to simpler equivalent inequalities.

   (a) $x + 12 < 39$

   (b) $\frac{5}{7}x < 36 - x$

   (c) $\sqrt{2} + 2x > 3\sqrt{2}$

   (d) $t\sqrt{3} < 3$

   (e) $8y - 3 > 3y + 7$

   (f) $\frac{t}{3} < 4 + \frac{t}{6} - 2$

   (g) $x^2 + 5 \geq 4$

   (h) $\frac{3}{x^2 + 4} < -2$

   (i) $-\frac{2}{x^2 + 2} \geq -1$

2. Solve the following sentences.

   (a) $1 < 4x + 1 < 2$
   (This is equivalent to "$1 < 4x + 1$ and $4x + 1 < 2$").

   (b) $4t - 4 < 0$ and $1 - 3t < 0$

   (c) $-1 < 2t < 1$

   (d) $6t + 3 < 0$ or $6t - 3 > 0$

   (e) $|x - 1| < 2$

   (f) $|2t| < 1$

   (g) $|x + 2| < \frac{1}{2}$

   (h) $|y + 2| > 1$

3. Graph the truth sets of the sentences in Problems 2(a), (c), (e) and (h) .

4. Determine which of the following are negative real numbers for every value of x.

   (a) x

   (b) $-x$

   (c) $\frac{1}{-x^2 - 1}$

   (d) $|-x - 1|$

   (e) $-|x + 1|$

   (f) -5

5. Solve $3y - x + 7 < 0$ for y; that is, obtain an equivalent sentence with y alone on the left side. What is the truth set for y when $x = 1$? Now solve $3y - x + 7 < 0$ for x. What is the truth set for x if $y = -2$?

6. If the area of a rectangle is 12 square inches and its length is less than 5 inches, what is its width?

7. Write an open sentence expressing that a certain negative number is less than its reciprocal. Solve the sentence.

13-3. Equations Involving Factored Expressions

When in Chapter 12 you solved quadratic equations of the form

$$(x - 3)(x + 2) = 0,$$

you needed the important property of numbers (Theorem 7-8e).

For real numbers a and b, $ab = 0$
if and only if $a = 0$ or $b = 0$.

Restate this property for the particular a and b in the above equation. Interpret the "if and only if" in your own words. It is this property, and the fact that $x - 3$ and $x + 2$ are real numbers for every real number x, that guarantee the equivalence of the sentence "$(x - 3)(x + 2) = 0$" and the sentence "$x - 3 = 0$ or $x + 2 = 0$". Thus, the truth set is $\{3, -2\}$.

How would you extend this property to equations such as $abcd = 0$? State a general property for any number of factors. What is the truth set of

$$(x + 1)(x - 3)(2x + 3)(3x - 2) = 0 \;?$$

Problem Set 13-3a

1. Find the truth sets of:
   (a) $(a + 2)(a - 5) = 0$
   (b) $(x + 3)(x + 1)(x - 2)(x) = 0$
   (c) $(3y - 1)(2y + 1)(4y - 3) = 0$

2. Solve:
   (a) $x^2 - x - 2 = 0$
   (b) $0 = x^2 - 121$
   (c) $(x^2 - 1)(x^2 + 5x + 6) = 0$
   (d) $(x^2 - 5)(x^2 - 24) = 0$
   (e) $x^3 = 25x$
   (f) $2x^2 - 5x = 3$
   (g) $x^3 + x = 2x^2$
   (h) $x^2 + 2 = 0$
   (i) $3x^2 = 21x - 18$
   (j) $x^2 - 4x + 2 = 0$
   (k) $x^2 + 6x = 1$
3. Solve $x^3 = 8$ by guessing a solution and showing it is the only real solution by arguing about the "size" of x.

   (If $x < 2$, what about x^3 ? If $x > 2$, what about x^3 ?)
4. Solve $x^4 = 1$ by writing it $(x^2)^2 - 1 = 0$ and factoring.
5. Find a polynomial which has the value 0 whenever x takes a value in the set $\{1, -1, 0\}$.
6. Find the truth set of the sentence

$$(x - 3)(x - 1)(x + 1) = 0 \text{ and } |x - 2| < 2 .$$

We have been careful to avoid adding or multiplying by an expression which for some value of the variable is not a real number. Let us look at another example which illustrates this danger.

Consider this example: Solve $(x - 3)(x^2 - 1) = 4(x^2 - 1)$.

Our first impulse is to multiply both sides by $\frac{1}{x^2 - 1}$. But for some values of x, $\frac{1}{x^2 - 1}$ is not a real number. Which values? Instead, since $4(x^2 - 1)$ _is_ a real number for every x, let us add $-4(x^2 - 1)$ to both members, giving

$$(x - 3)(x^2 - 1) - 4(x^2 - 1) = 0$$

$$(x - 3 - 4)(x^2 - 1) = 0 \qquad \text{(Why?)}$$

$$(x - 7)(x - 1)(x + 1) = 0$$

Each of these sentences is equivalent to every other. What is the resulting truth set? If we had multiplied each side (unthinkingly) by $\frac{1}{x^2 - 1}$, what would be the truth set of the resulting sentence?

This example warns us that "$ac = bc$" and "$a = b$" are _not_ equivalent. Instead, we follow a sequence of _equivalent_ sentences:

$$ac = bc$$

$$ac - bc = 0$$

$$(a - b)c = 0$$

$$a - b = 0 \text{ or } c = 0$$

This tells us that the sentence

$$ac = bc$$

is equivalent to the sentence

$$a - b = 0 \text{ or } c = 0,$$

when a, b, and c are real numbers.

Problem Set 13-3b

1. Solve

   (a) $x(2x - 5) = 7x$

   (b) $(3 + x)(x^2 + 1) = 5(3 + x)$

   (c) $(x - 2)(3x + 1) = (x - 2)(x - 5)$

   (d) $3(x^2 - 4) = (4x + 3)(x^2 - 4)$

   (e) $5x - 15 = x^2 - 3x$

2. Multiply both members of the equation "$x^2 = 3$" by $(x - 1)$. Are the new and the original truth sets the same? Is $x - 1$ zero for some value of x ?

3. Multiply both members of the equation "$t^2 = 1$" by $(t + 1)$. Compare the new and the original truth sets. Discuss any differences the two multiplications made in the truth sets in Problems 2 and 3.

13-4. Fractional Equations

The expression $\frac{1}{x}$ is not a real number when x is 0. Therefore, when we try to solve the equation

$$\frac{1}{x} = 2$$

we are limited to numbers other than 0. In other words, we must solve the sentence

$$\frac{1}{x} = 2 \text{ and } x \neq 0 .$$

Knowing that x cannot be 0, we may then multiply by the non-zero number x to obtain

$$\frac{1}{x} \cdot x = 2x \text{ and } x \neq 0,$$

$$1 = 2x \text{ and } x \neq 0.$$

Hence, "$\frac{1}{x} = 2$ and $x \neq 0$" and "$1 = 2x$ and $x \neq 0$" are equivalent sentences. The latter has the truth set $\{\frac{1}{2}\}$. Thus $\frac{1}{2}$ is the solution.

Another way to handle this same problem is to add -2 to both members of "$\frac{1}{x} = 2$", giving

$$\frac{1}{x} - 2 = 0$$

$$\frac{1 - 2x}{x} = 0 \qquad \text{(Why?)}$$

What are the requirements on a and c for the number $\frac{a}{c}$ to be 0 ? They are, first, that $c \neq 0$ (Why?), and second, that $a = 0$ (Why?). Thus the sentence "$\frac{a}{c} = 0$" is equivalent to the sentence "$a = 0$ and $c \neq 0$".

Then "$\frac{1 - 2x}{x} = 0$" is equivalent to what sentence? Your answer should be "$1 - 2x = 0$ and $x \neq 0$", which is the same sentence we had before. Can you find the truth set of "$\frac{x + 1}{x - 2} = 0$" in the same way.

The same two approaches can be used on more complicated fractional equations. Thus we can solve the equation

$$\frac{1}{x} = \frac{1}{1 - x}$$

either by multiplying both members by a suitable polynomial (What is it?), or by writing it first as $\frac{1}{x} - \frac{1}{1 - x} = 0$ and then simplifying to a single fraction. In either case we must recognize two "illegal values" for x, 0 and 1 . The solution is subject to x not taking on those values. Using the second method, we get "$\frac{(1 - x) - x}{x(1 - x)} = 0$" which is equivalent to "$1 - 2x = 0$ and $x \neq 0$ and $x \neq 1$". The solution of this sentence is $\frac{1}{2}$, which is, therefore, the solution of the original sentence.

As a final example, solve

$$\frac{x}{x - 2} = \frac{2}{x - 2} .$$

Since $x \neq 2$, then upon multiplying both members by $x - 2$ we obtain

$$\frac{x}{x - 2} \cdot (x - 2) = \frac{2}{x - 2} \cdot (x - 2) \quad \text{and} \quad x \neq 2,$$

$$x = 2 \quad \text{and} \quad x \neq 2.$$

Hence the sentence "$\frac{x}{x - 2} = \frac{2}{x - 2}$" is equivalent to the sentence "$x = 2$ and $x \neq 2$". What is the truth set of this sentence?

Problem Set 13-4

Solve the following equations.

1. $\frac{2}{x} - \frac{3}{x} = 10$
2. $\frac{x}{2} - \frac{x}{3} = 10$
3. $x + \frac{1}{x} = 2$
4. $y - \frac{2}{y} = 1$
5. $\frac{s - 2}{s} + \frac{3}{s^2} = 1$
6. $\frac{3}{2y} - \frac{2 + 5y}{y} = \frac{1}{3}$
7. $\frac{1}{y} - \frac{1}{y - 4} = 1$
8. $\frac{1}{t} = \frac{1}{t - 1}$
9. $\left(\frac{x - 1}{x + 1}\right)^2 = 4$
10. $\frac{-2}{x - 2} + \frac{x}{x - 2} = 1$
11. $\left(\frac{x}{x + 1}\right)(x^2 - 1) = 0$
12. $\frac{t}{1 + t} + \frac{t}{t - 1} = 0$
13. $\frac{1 - y}{1 + y} + \frac{1 + y}{1 - y} = 0$
14. $\frac{1 - y}{1 + y} - \frac{1 + y}{1 - y} = 0$

*15. $\frac{1}{y} + \frac{2}{1 - y} + \frac{1}{1 + y} = 0$

*16. $\frac{1}{x} + \frac{1}{1 - x} + \frac{1}{1 + x} = 0$

17. The sum of a number and its reciprocal is -2. What is the number?
18. (a) Printing press A can do a certain job in 3 hours and press B can do the same job in 2 hours. If both presses work on the job at the same time, in how many hours can they complete it?

(b) If presses A and C work on the job together and complete it in 2 hours, how long would it take press C to do the job alone?

(c) Presses A and B begin the job, but at the end of the first hour press B breaks down. If A finishes the job alone, how long does A work after B stops?

19. In each of the following, express the indicated variable in terms of the others.

Example: $V = \frac{1}{3}Bh$; B .

This is equivalent to

$$V\left(\frac{3}{h}\right) = \frac{1}{3}Bh\left(\frac{3}{h}\right) \text{ and } h \neq 0,$$

that is, to

$$\frac{3V}{h} = B \text{ and } h \neq 0.$$

(a) $A = \frac{1}{2}bh$; h

(b) $T = \frac{D}{R}$; R

(c) $A = \frac{1}{2}h(x + y)$; h

(d) $S = \frac{n}{2}(a + \ell)$; ℓ

(e) $\frac{1}{a} + \frac{1}{b} = 1$; b

13-5. Squaring

If $a = b$, then of course $a^2 = b^2$. Why? Do you think it is true, conversely, that if $a^2 = b^2$ then $a = b$? You may see at once that this is not so. Give an example. Hence, "$a^2 = b^2$" and "$a = b$" are not equivalent sentences.

On the other hand, we can alter $a^2 = b^2$ through a chain of equivalent sentences as follows:

$$a^2 = b^2$$

$$a^2 - b^2 = 0$$

$$(a - b)(a + b) = 0$$

$$a - b = 0 \text{ or } a + b = 0$$

$$a = b \text{ or } a = -b$$

Tell why each of these sentences is equivalent to the next one. Thus, "$a^2 = b^2$" and "$a = b$ or $a = -b$" are equivalent sentences.

If we square both members of the sentence "$x = 3$", we obtain "$x^2 = 9$", which is equivalent to "$x = 3$ or $x = -3$". Thus, squaring the members of a sentence sometimes enlarges the truth set.

Problem Set 13-5a

Tell what squaring both members does to the truth sets of the following equations.

1. $x = 2$
2. $x - 1 = 1$
3. $x + 2 = 0$
4. $x - 1 = 2$

In the above problems it is obvious what the original truth set is, and we haven't had to use the new truth set to obtain the old. However, sometimes we square both members of an equation as a simplifying process in situations where we don't already know the truth set. We do know, as in the above problems, that any solution of the original equation is a solution of the equation obtained by squaring. But we also know that the new truth set may be larger than the old. Therefore, each solution of the new equation must be checked in the original equation in order to eliminate any possible extra solutions that may have crept in during the squaring.

Example 1. Solve $\sqrt{x + 3} = 1$.

If $\sqrt{x + 3} = 1$ is true for some x,

then $(\sqrt{x + 3})^2 = (1)^2$ is true for the same x;

$$x + 3 = 1,$$

$$x = -2.$$

If $x = -2$, then $\sqrt{x + 3} = \sqrt{-2 + 3} = \sqrt{1} = 1$.

Hence, -2 is the solution.

Example 2. Solve $\sqrt{x} + x = 2$.

Our objective is to square both members and obtain an equation free of radicals. Let us try it.

$$(\sqrt{x} + x)^2 = 2^2$$

$$(\sqrt{x})^2 + 2(\sqrt{x})(x) + x^2 = 2^2$$

$$x + 2x\sqrt{x} + x^2 = 4.$$

Apparently, we have arrived at a more complicated sentence which still contains a radical. Instead, let us write the sentence in the equivalent form

$$\sqrt{x} = 2 - x$$

before squaring its members. Then we obtain

$$(\sqrt{x})^2 = (2 - x)^2$$

$$x = 4 - 4x + x^2$$

$$0 = 4 - 5x + x^2$$

$$0 = (x - 4)(x - 1)$$

$$x = 4 \quad \text{or} \quad x = 1.$$

In other words, if there are solutions of the sentence they must be in the set $\{1, 4\}$. Checking each of these possibilities, we find that 4 does not make the original sentence true, while 1 does. The solution is therefore 1.

Example 3. Solve $|x| - x = 1$.

Again we can obtain a simpler sentence by squaring. Here we use a fact about absolute values which you should prove in Problem 11: $|x|^2 = x^2$ for every real number x. Then we have the sequence of sentences

$$|x| = x + 1$$

$$(|x|)^2 = (x + 1)^2$$

$$x^2 = x^2 + 2x + 1$$

$$2x + 1 = 0$$

$$x = -\frac{1}{2}$$

Checking back, we find that $-\frac{1}{2}$ does make the original equation true and is, therefore, its solution.

Problem Set 13-5b

Solve the following equations by squaring.

1. $\sqrt{2x} = 1 + x$
2. $\sqrt{2x + 1} = x + 1$
3. $\sqrt{x + 1} - 1 = x$
4. $\sqrt{4x} - x + 3 = 0$
5. $3\sqrt{x + 13} = x + 9$
6. $|2x| = x + 1$
7. $2x = |x| + 1$
8. $x = |2x| + 1$
9. $x - |x| = 1$
10. $|x - 2| = 3$
11. Prove: For every real number x,

$$|x|^2 = x^2 .$$

12. The distance between x and 3 on the number line is 2 more than x. Solve for x.
13. One leg of a right triangle is 8 inches long and the hypotenuse is 4 inches less than the sum of the two legs. Find the other leg.
14. The time t in seconds it takes a body to fall from rest a distance of s feet is given by the formula $t = \sqrt{\frac{2s}{g}}$. Find s if $t = 6.25$ seconds and $g = 32$.
15. Using the formula in Problem 14, find a formula for g in terms of t and s.

16. Determine whether each of the following pairs of sentences in two variables are equivalent.

(a) $x^2 + y^2 = 1$, $y = \sqrt{1 - x^2}$

(b) $x^2 + y^2 = 1$, $\sqrt{x^2 + y^2} = 1$

(c) $x^2 = xy$, $x = 0$ or $x = y$

13-6. *Polynomial Inequalities

Is $(-4)(3)(5)(-6)(-8)$ a positive number? A negative number? Did you need to perform the multiplication to answer this question?

When we multiply several non-zero numbers together, their product is positive if the number of negative factors is even, and their product is negative if the number of negative factors is odd.

This means that we can tell immediately whether a factored polynomial, such as

$$(x + 3)(x + 2)(x - 1),$$

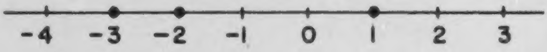
is positive, negative, or 0 for any given x . How about this polynomial for $x = 2$? For $x = 0$? For $x = -1$? For $x = -\frac{5}{2}$? For $x = -4$? You need not compute the value of the polynomial; just check how many factors are negative.

We can do better than choosing a few points at random. We can first find the set of values of x for which

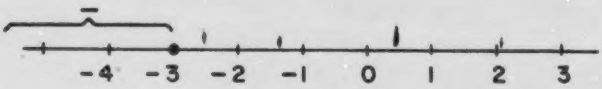
$$(x + 3)(x + 2)(x - 1) \text{ is } 0$$

$$(\text{the truth set of } (x + 3)(x + 2)(x - 1) = 0).$$

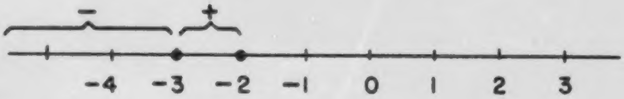
What is this set? Then we draw the graph of this set on the number line.



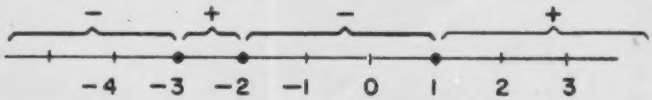
What can we say about each of the factors $(x + 3)$, $(x + 2)$, $(x - 1)$ for any x less than -3? Try $x = -4$. We find that all three factors are negative numbers, and therefore their product is negative. We indicate this on the number line as follows.



What about these factors when x is between -3 and -2? Try $x = -\frac{5}{2}$. The factor $(x + 3)$ is now positive, while the other two remain negative. We can think of $(x + 3)$ as "changing over" from negative to positive as x crosses -3. The product is now positive for x between -3 and -2. We indicate this by a mark "+" over the interval.



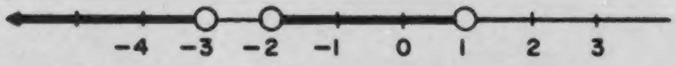
Probably you can see what is going to happen when x crosses -2 and finally 1. When x crosses -2, the next factor $(x + 2)$ changes from negative to positive, so that for any x between -2 and 1 there are two positive factors and one negative, the product now being negative. Finally, when x crosses 1, the last factor changes from negative to positive, so that for x greater than 1, all factors are positive.



Using this final diagram, we can read off the truth sets of certain related inequalities. For example, the truth set of the sentence

$$(x + 3)(x + 2)(x - 1) < 0$$

is graphed on page 400. This is the set of all numbers x for



which the product of the factors is negative, namely, the set of all x such that

$$x < -3 \quad \text{or} \quad -2 < x < 1 .$$

What is the truth set of the sentence $(x + 3)(x + 2)(x - 1) > 0$? Draw its graph. Of the sentence $(x + 3)(x + 2)(x - 1) \geq 0$?

To find the truth set of

$$x^2 - 3 \leq 2x ,$$

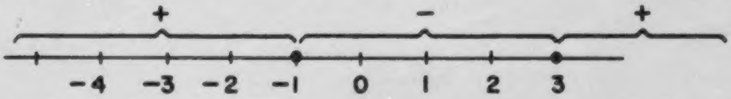
we first change to the equivalent inequality with 0 on the right side:

$$x^2 - 2x - 3 \leq 0 .$$

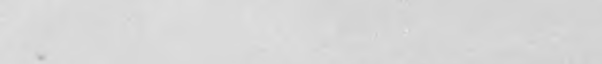
Then we factor the left side into first degree polynomial factors:

$$(x + 1)(x - 3) \leq 0 .$$

Proceeding as before, we get the diagram.



Thus, the truth set of the inequality $x^2 - 3 \leq 2x$ has the following graph (since the product of the factors $(x + 1)(x - 3)$ must be negative or zero).



This is the set of all x such that $-1 \leq x \leq 3$.

Problem Set 13-6a

1. Using the above discussion as a model, draw the graph and describe the truth sets of the following inequalities.

   (a) $(x - 1)(x + 2) > 0$

   (b) $y^2 < 1$

   (c) $t^2 + 5t \leq 6$

   (d) $x^2 + 2 \geq 3x$

   (e) $(s + 5)(s + 4)(s + 2)(s)(s - 3) < 0$

   (f) $2 - x^2 < x$

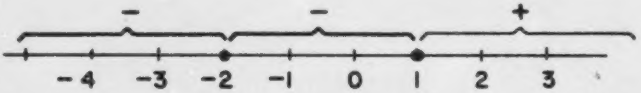
2. What is the truth set of the sentence

$$(x + 2)(x - 1) > 0 \quad \text{and} \quad x < 3 \; ?$$

*3. Is there a single polynomial inequality equivalent to the sentence of Problem 2?

There is one danger point which we should explain.

Suppose a factor is repeated in a polynomial one or more times, as in $(x + 2)^2(x - 1)$. When x crosses -2, there are two factors of the polynomial changing together from negative to positive. The number of negative factors drop from 3 to 1, and the product remains negative as x crosses -2. The diagram is then



What is the truth set of

$$(x + 2)^2(x - 1) > 0 \; ?$$

(i.e., for what values of x is the product of the factors positive?) Notice that the truth set of

$$(x + 2)^2(x - 1) \geq 0$$

is the set of all x such that $x \geq 1$ or $x = -2$.

What happens if a factor occurs three times, as in $x(x - 1)^3$? What is the truth set of

$$x(x - 1)^3 < 0 \text{ ?}$$

Of

$$x(x - 1)^3 \geq 0 \text{ ?}$$

Sometimes we have a quadratic factor, such as $x^2 + 2$, which cannot be factored and which is always positive for all values of x . Does such a factor have any effect on the way the product changes from positive to negative? What is the truth set of

$$(x^2 + 2)(x - 3) < 0 \text{ ?}$$

Of

$$(x^2 + 2)(x - 3) \geq 0 \text{ ?}$$

Problem Set 13-6b

Solve and graph:

1. $x^2 + 1 > 2x$
2. $x^2 + 1 < 0$
3. $(t^2 + 1)(t^2 - 1) \geq 0$
4. $4s - s^2 > 4$
5. $(x - 1)^2(x - 2)^2 > 0$
6. $(y^2 - 7y + 6) \leq 0$
7. $(x + 2)(x^2 + 3x + 2) < 0$
8. $3y + 12 \leq y^2 - 16$
9. $x^2 + 5x > 24$
10. $|x|(x - 2)(x + 4) < 0$

Review Problems

For each pair of sentences in Problems 1 - 6, determine whether the two sentences are equivalent.

1. $x(x^2 + 1) - 3(x^2 + 1) = 0$, $\qquad x - 3 = 0$

2. $(x - 3) \cdot \frac{x^2 - 1}{x^2 - 1} = 2$, $\qquad x - 3 = 2$

3. $(x - 3)(x^2 + 4) = 2(x^2 + 4)$, $\qquad x - 3 = 2$

4. $\frac{x}{x - 3} - \frac{3}{x - 3} = 0$, $\qquad x = 3$

5. $\frac{x + 2}{x - 2} = 0$, $\qquad |x| = 2$

6. $\frac{|x|}{2} = 1$, $\qquad x^2 = 4$

In Problems 7 - 20, find the truth set of

7. $\frac{x}{x - 3} - \frac{3}{x - 3} = 0$

8. $\frac{x}{x - 3} + \frac{3}{x - 3} = 0$

9. $(x + 1)(x - 3) = 7(x - 3)$

10. $(x + 1)(x^2 - 2) = -(x + 1)$

11. $x(x - 1)(x - 2) = 0$

12. $\frac{2}{x - 8} = \frac{1}{x - 2}$

13. $\frac{1}{x} - 3 + \frac{2x - 1}{x} = 0$

14. $\sqrt{x + 2} + 2 = 0$

15. $\sqrt{x + 2} - 2 = 0$

16. $|x + 1| = 3$

17. $|x| + x = 1$

18. $|x| + 1 = x$

19. $\frac{x + 1}{x + 1} = 1$

20. $\frac{x^2 + 1}{x^2 + 1} = 1$

21. Solve and graph the following sentences.

(a) $\frac{x + 2}{x - 2} = 0$

(b) $\frac{x + 2}{x - 2} > 0$

(c) $x^2 - 4 > 0$

(d) $|x| > 2$

22. Solve and graph

$$\sqrt{1 + 2x} < x - 1 .$$

23. Graph the truth set of each of the following sentences.

(a) $(x - 3)(x - 1)(x + 1) > 0$

(b) $(x - 3)(x - 1)(x + 1) > 0$ and $x \geq 0$

(c) $(x - 3)(x - 1)(x + 1) > 0$ or $x \geq 0$

Chapter 14

GRAPHS OF OPEN SENTENCES IN TWO VARIABLES

14-1. The Real Number Plane

The real number line has been helpful in making decisions about relations among real numbers. (Give examples of some cases in which we have used the real number line.) Perhaps a real number plane would be even more helpful.

We have associated numbers with points of the line. How can we associate numbers with points of the plane? Consider any point P in a plane. If this point is on the number line x units from the zero point, then there is one number x associated with P. If P is not on the number line, as in Figure 1, there is a number, in this case 7, which is associated with P in the sense that it is on the number line directly under P (it would be directly over P if P were below the number line). This number is not sufficient by itself to locate P, since 7 is on the number line and P is 3 units above the line. Perhaps we need two numbers to name P, using 7 as the first one because it is on the number line which we already know. From 7 we could draw a second

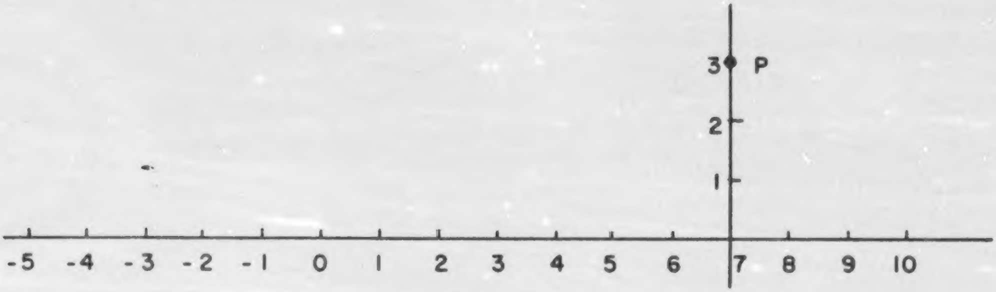


Figure 1.

number line to P, drawing it vertically and with the zero of this line lying on the first line. On this vertical line we find P at the number 3; this number we regard as the second number belonging to P. The point P now has the associated pair of numbers (7, 3). We write these as indicated, placing the number found along the horizontal line first, and the one found along the vertical line second, and enclosing them in parentheses. We have now assigned to P a first number, 7, and a second number, 3, and we think of these as an ordered pair of numbers, (7, 3), belonging to P and called the coordinates of P. The first number is called the abscissa of P and the second number the ordinate of P.

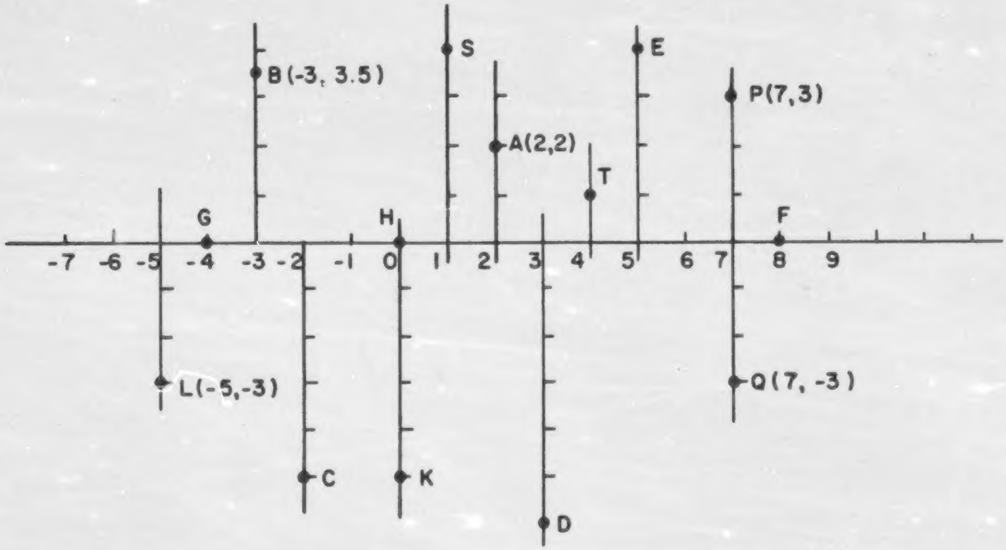


Figure 2.

In Figure 2, verify that the ordered number pairs written for point P, A, B, L, and Q are correct. Which number is written first in each case? Which is second? The order is important, and hence we consider only ordered pairs. Note that the ordered pair (1, 4) is not the same as the ordered pair

(4, 1). How does the ordered pair for Q differ from that for P ? Why is the second number for Q negative?

In Figure 2, what ordered pairs of numbers are associated with points E, C, K, and D ? What ordered pair is associated with point H, which is on the number line? This point is called the origin, and is associated with (0, 0). We may consider that the second number of the ordered pair is the distance above or below the number line. What ordered pairs are associated with points F and G ? Make a general statement about the second number of the ordered pair associated with any point which lies on the horizontal number line.

If we have several points in a plane, and a single horizontal number line, is there any way in which we can refine our figure so that we can identify the ordered pairs of numbers associated with the points without drawing a separate vertical line from each point to the horizontal number line? For this purpose we shall use coordinate paper and draw only one second number line, passing through the origin and perpendicular to the first number line. If we label the units of measure on both of these lines, each of which is called a coordinate axis, the network of the coordinate paper will permit us to choose quickly the suitable numbers of an ordered pair.

Note that S and T in Figure 2 do not have the same coordinates; the first coordinate of S is 1 but the first coordinate of T is 4. The coordinates of S are the ordered number pair (1, 4), while those of T are the ordered number pair (4, 1). The same numbers appear in each pair, but since the order is different, the ordered pairs are different.

Do you think it is always true that two different points in the plane cannot have the same coordinates?

Problem Set 14-1a

1. Write the ordered pairs of numbers which are associated with the points A through M in the figure below:

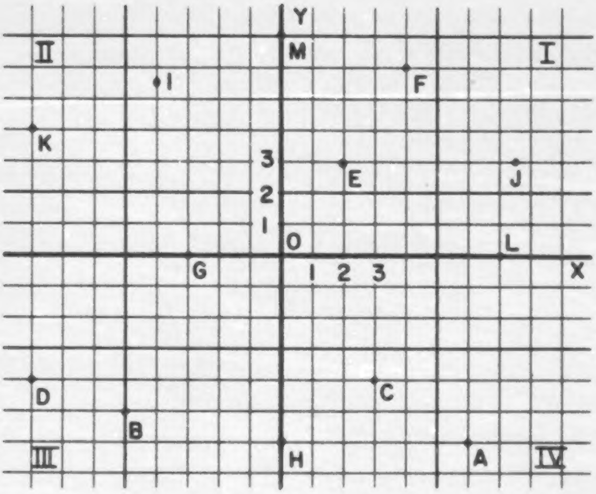


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Figure for Problem 1.

2. The four parts of the plane into which the number lines divide the plane are called quadrants. These quadrants are numbered counter-clockwise, as in the figure for Problem 1 beginning with the quadrant in the upper right-hand corner where both coordinates are positive. In which quadrants will the points lie for which the second coordinate is equal to the first?

3. Locate four points whose ordered pairs have ordinates -3? Consider all such points; this set of points forms what sort of figure?

4. Where are all the points whose ordered pairs have abscissas $\frac{3}{2}$? Describe this set of points.

Now let us shift our attention from points to numbers. Select any real number as the first number of a pair, and then select any real number as the second of the pair. Can we associate with each such ordered pair a point on the plane? Given the ordered pair $(-3, \frac{2}{3})$, can you find a point of the plane having these coordinates? Explain how you locate the point. Does every ordered pair correspond to at least one point? Exactly one point?

Is it clear that for each point of the plane there is exactly one ordered pair of real numbers and for each ordered pair there is exactly one point?

Problem Set 14-1b

1. Using coordinate axes of your own choosing, locate the points associated with the following ordered pairs of numbers.

   $A(1, -3)$

   $B(-6, 4)$

   $C(0, \frac{8}{3})$

   $D(-7, -1)$

   $E(-4, 0)$

   $F(0, 0)$

   $G(5, \frac{3}{2})$

   $H(\frac{3}{2}, 5)$

   $I(-4, -6)$

   $J(-6, -4)$

   $K(0, -\frac{5}{3})$

   $L(-\frac{5}{3}, 0)$

2. In the figure you have just drawn for Problem 1, are points G and H the same? Why? Are points I and J the same? K and L ?

3. With reference to a set of coordinate axes, mark the points with coordinates: $(2, 3)$, $(2, 1)$, $(2, \frac{1}{2})$, $(2, 0)$, $(2, -5.5)$, $(2, -\frac{7}{2})$. What is true of all of these ordered pairs of numbers? Describe the set of all the points for which the abscissa of the ordered pair is 2 .

4. If you locate several points whose ordered pairs have 5 for their ordinates, where would all these points lie?

5. With reference to a set of coordinate axes, locate eight points whose coordinates are pairs of numbers for which the first and second numbers are the same. If you could locate all such points, what sort of figure would you have?

*6. Let us think of moving all the points of a plane in the following manner: Each point with coordinates (c, d) is moved to the point with coordinates (-c, d). Describe this in terms of taking the opposite. Another way of looking at this is to consider that the points of the plane are rotated one-half a revolution about the y-axis, as indicated in the figure for the problem. Answer the following questions, and locate the points referred to in parts (a) and (b).

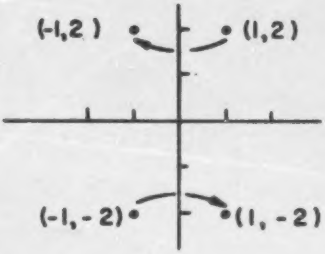

Figure for Problem 6.

(a) To what points do the following points go:

$(2, 1)$, $(2, -1)$, $(-\frac{1}{2}, 2)$, $(-1, -1)$, $(3, 0)$,

$(-5, 0)$, $(0, 2)$, $(0, -2)$?

(b) What points go to the points listed in (a) above?

(c) What point does $(c, -d)$ go to?

(d) What point does $(-c, d)$ go to?

(e) What point goes to (c, d)?

(f) What points go to themselves?

*7. Suppose a point with coordinates (c, d) is moved to the point $(c + 2, d)$. This can be thought of as sliding the points of the plane to the right 2 units. Answer the following questions and locate all of the points in parts (a) and (b).

(a) What points do the following points go to:
$(1, 1)$, $(-1, 1)$, $(-2, 2)$, $(0, -3)$, $(3, 0)$?

(b) What points go to the points listed in (a) above?

(c) To what point does $(c - 2, d)$ go?

(d) What point goes to $(-c, d)$?

(e) Which points go to themselves?

14-2. Graphs of Open Sentences With Two Variables

If we assign the values 0 and -2 to the variables of the open sentence

$$3y - 2x + 6 = 0 ,$$

is it then a true sentence? To which variable did you assign 0 ? To which -2 ? Were there two different ways to make these assignments?

To avoid the kind of confusion you met in the preceding paragraph, let us agree that whenever we write an open sentence with two variables, we must indicate which of the variables is to be considered first. When the variables are x and y, as in the above example, x will always be considered first.

With this agreement we are ready to examine the connection between an open sentence with two ordered variables and an ordered pair of real numbers. Among the set of all ordered pairs of real numbers, each pair has a first number which we associate with the first variable and a second number which we associate with the second variable. In this way an open sentence with two ordered variables acts as a sorter--it sorts the set of all ordered pairs of real numbers into two subsets:

(1) the set of ordered pairs which make the sentence true, and (2) the set of ordered pairs which make the sentence false. As before, we call this first set the truth set of the sentence.

Now we can answer the question in the first paragraph if we specify the ordered pair (0, -2). Does the ordered pair (0, -2) belong to the set of ordered pairs for which

$$3y - 2x + 6 = 0$$

is a true sentence? Does the ordered pair (-2, 0) belong to the truth set?

An ordered pair belonging to the truth set of a sentence with two variables is called a solution of the sentence, and this ordered pair is said to satisfy the sentence. If r is taken as the first variable, what are some solutions of

$$s = r + 1 \ ?$$

Does the ordered pair (-2, -3) satisfy this sentence? Is (-3, -2) a solution?

If u is taken as the first variable, what are some ordered pairs in the truth set of

$$v = 2u^2 \ ?$$

Is (-1, 2) a solution of this sentence? Does (2, -1) satisfy this sentence?

Throughout this chapter we shall use only x and y as variables, in order to focus attention on properties of sentences with two ordered variables. But many times in the future you will see other variables used, and then you must always decide which variable is used first.

One other point needs to be stressed. The sentence "$y = 4$" can be considered as a sentence in one variable y, or it can be considered as a sentence with two ordered variables x and y. When we say that "$y = 4$" is a sentence with two variables, we mean that "$y = 4$" is an abbreviation for

$$(0)x + (1)y = 4 \ .$$

What are some solutions of this sentence? What is true of every ordered pair satisfying this sentence? If "$x = -2$" is a sentence with two variables, then "$x = -2$" is an abbreviation for

$$(1)x + (0)y = -2 .$$

What is true of every ordered pair satisfying this sentence?

Problem Set 14-2a

1. Describe the truth sets of the following open sentences in two ordered variables x, y.

   (a) $y = 5$
   (b) $x = 0$
   (c) $y = -3x$
   (d) $x = 3$

2. Find four solutions of each of the following open sentences.

   (a) $y = 3x - 2$
   (b) $y = 2 + x$
   (c) $y = x^2 + 1$
   (d) $y = |x|$

3. For each of the sentences in Problem 2, find two ordered pairs which do not satisfy the sentence.

4. For each sentence in Problem 2 locate on a set of coordinate axes the points whose coordinates are the solutions you found. Use a separate set of coordinate axes for each part.

Remember that every point of the plane has an associated pair of numbers called its coordinates. Now we see that an open sentence with two ordered variables not only sorts the set of ordered pairs of numbers into two subsets--it also sorts the points of the plane into two subsets:

(1) The set of all points whose coordinates satisfy the sentence, and
(2) all other points.

As before, we call this first set of points the graph of the sentence.

We shall be interested to learn what sort of figure on the plane this graph will be for any given sentence. Let us try, as an example, the sentence

$$2x - 3y - 6 = 0 .$$

We can guess several solutions, such as (3, 0) and (0, -2). Try to guess some more solutions. Notice that it would be easier to determine solutions if we write an equivalent equation having y by itself on the left side:

$$\begin{aligned} 2x - 3y - 6 &= 0 , \\ -3y &= -2x + 6 , \\ 3y &= 2x - 6 , \\ y &= \frac{2}{3}x - 2 . \end{aligned}$$

We call this last equivalent sentence the y-form of the original sentence. Now we see that "$y = \frac{2}{3}x - 2$" can be translated into an English sentence in terms of abscissas and ordinates of points on its graph: "The ordinate is 2 less than $\frac{2}{3}$ of the abscissa."

Since we shall be taking $\frac{2}{3}$ of the abscissa, it is easiest to calculate ordinates corresponding to abscissas which are multiples of 3. If the abscissa is 3, the ordinate must be 0 in order to form a solution. Why? If the abscissa is -6, what must the ordinate be? Continuing, we can fill in a table of ordered pairs which satisfy the sentence:

x	-9	-6	-3	0		5	
y		-6		-2	0		9

You fill in the empty squares.

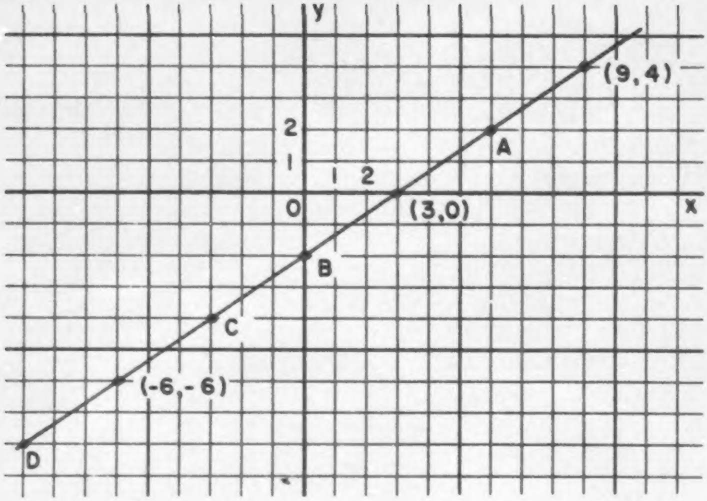

Figure 3.

In Figure 3, the points (-6, -6), (3, 0) and (9, 4) seem to lie on a straight line. Do the other points whose coordinates you found in the table of solutions also seem to lie on this line? This brings up the question: If we draw the line through these points, will we find on it all the points such that each has "ordinate 2 less than $\frac{2}{3}$ of the abscissa?" Furthermore, we must ask: Is every point on this line a point whose "ordinate is 2 less than $\frac{2}{3}$ of the abscissa"?

Suppose we try a point which appears to be on the line, such as point A in Figure 3. The coordinates of this point are (6, 2). Do these coordinates satisfy the equation $2x - 3y - 6 = 0$?

It turns out that every point on this line has coordinates which satisfy the equation $2x - 3y - 6 = 0$.

When we say that a specified line is the graph of a particular open sentence, we mean that both our questions above are answered affirmatively:

(1) if two ordered numbers satisfy the sentence, they are the coordinates of a point on the line;

(2) if a point is on the line, its coordinates satisfy the open sentence.

Thus, the line drawn in Figure 3 is the graph of the sentence

$$2x - 3y - 6 = 0 .$$

We can do the same with such open sentences as $3y + 5x - 11 = 0$, $2x + 5 = 0$, $-8y + 1 = 0$, etc., and in each case conclude that the graph is a line.

This suggests that the following general statement is true.

> If an open sentence is of the form
> $$Ax + By + C = 0 ,$$
> where A, B, C are real numbers with A and B not both zero, then its graph is a line; every line in the plane is the graph of an open sentence of this form.

We know that every open sentence has a graph, and we suspect that every graph is associated with an open sentence. Of course, some open sentences may have graphs with no points (empty graphs) and others with graphs which cover regions of the plane. Later we shall study such sentences.

Problem Set 14-2b

1. Where are all the points in the plane whose ordinates are -3?

2. With reference to one set of coordinate axes, locate the set of points whose coordinates satisfy the following equations, that is, the points whose pairs of coordinates belong to the truth sets of the equations.

   (a) $y = 5$
   (b) $y = 0$
   (c) $x = 0$
   (d) $x = -2$

What is an equation whose graph is the horizontal axis? What is an equation whose graph is the vertical axis?

3. With reference to a set of coordinate axes, find the points such that
   (a) each has the abscissa equal to the opposite of the ordinate, using all possible pairs of real numbers which have meaning within the scope of your graph. With reference to the same axes, locate
   (b) the points such that each has ordinate twice the abscissa;
   (c) the points such that each has ordinate that is the opposite of twice the abscissa.

   What general statements can you make concerning these graphs? Write open sentences for each of the graphs drawn.

4. With reference to one set of coordinate axes, draw the graphs of the following.

(a) $y = 3x$	(d) $y = -3x$
(b) $y = 6x$	(e) $y = -6x$
(c) $y = \frac{1}{2}x$	(f) $y = -\frac{1}{2}x$

   What characteristic do all of these have in common? How does the graph of (a) differ from the graph of (d) ? Does the same pattern apply to the graphs of (b) and (e) ? To the graphs of (c) and (f) ?

5. With reference to one set of coordinate axes, draw the graphs of the following, and label each one.

(a) $y = x + 5$	(d) $y = 2x - 5$
(b) $y = x - 3$	(e) $y = \frac{1}{3}x + 2$
(c) $y = 2x + 5$	(f) $y = -\frac{1}{3}x - 2$

   How does the graph of (a) differ from the graph of (b) ?
   How does the graph of (c) differ from the graph of (d) ?
   How does the graph of (e) differ from the graph of (f) ?
   What is true of the graphs of (a) and (b), and also of the graphs of (c) and (d), that is not true of the graphs of (e) and (f) ?

With respect to a set of coordinate axes locate points for which the abscissas are

$$-2, -1, 0, 1, 2, 3$$

respectively, and for which each ordinate is equal to 3 times the abscissa. Do these points lie on a line?

Now locate points having these same abscissas, but for which each ordinate is greater than 3 times the abscissa. Do these new points lie on a line? Does each one lie above the corresponding point of the first set?

The points in the first set satisfy the sentence

$$y = 3x$$

while those in the second set satisfy the sentence

$$y > 3x .$$

The sentence "$y = 3x$" is the equation of a line, and the graph of "$y > 3x$" is the set of all points above this line, as shown by the shaded portion of Figure 5. Thus, the graph of a sentence such as "$y > 3x$" is the set of all points of the plane for which the sentence is true. If the verb is "is greater than or equal to", that is "$\geq$", we make the boundary line solid, as in Figure 4, while the verb "is greater than" is indicated by using a dashed line for the boundary between the shaded and the unshaded regions as in Figure 5. In these two illustrations, the line is the graph of the sentence $y = 3x$. This graph, which is a line, separates the plane into two half-planes. The graph of $y < 3x$ is the half-plane such that every ordinate is less than three times the abscissa; it is the set of points below the line $y = 3x$. The graph of $y \leq 3x$ is the lower half-plane including the line $y = 3x$.

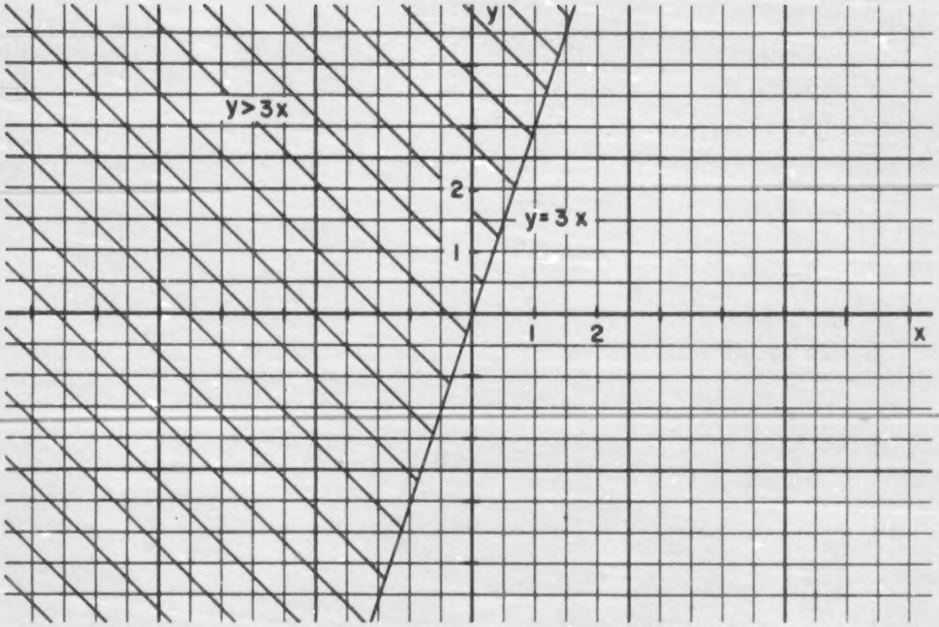

Figure 4. $y \geq 3x$

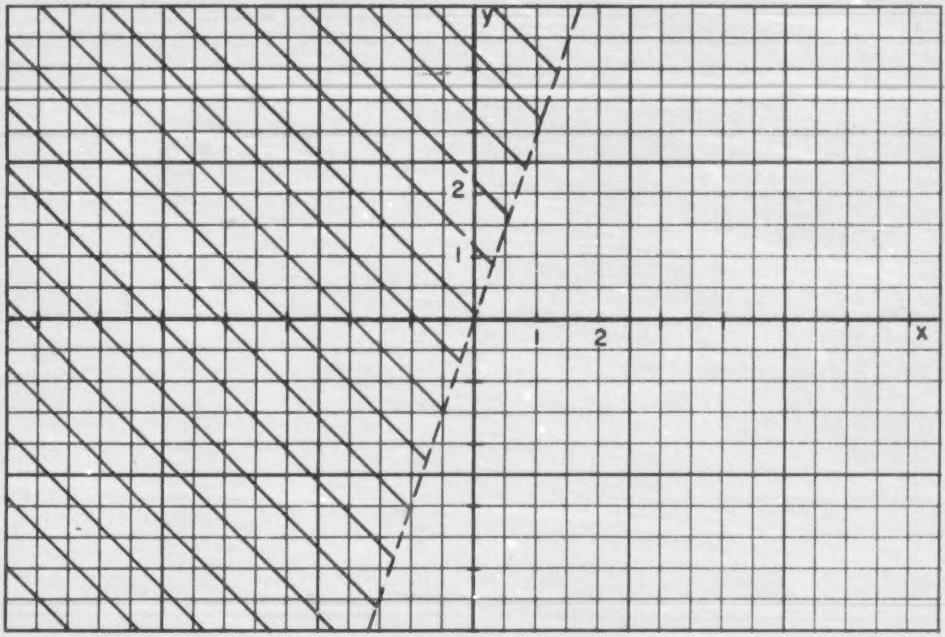

Figure 5. $y > 3x$

[sec. 14-2]

Problem Set 14-2c

1. With reference to a set of coordinate axes indicate the set of points associated with the ordered pairs of numbers such that each has an ordinate two greater than the abscissa. What open sentence can you write for this set? Now draw the graph of the following open sentences.

   (a) $y > x + 2$; (b) $y \geq x + 2$.

   Is it possible to draw both of these graphs with reference to the same coordinate axes?

2. Given $y = |x|$. In this sentence, is y ever negative? Write the solutions for which the abscissas are: -3, -1, 0, $1\frac{1}{2}$, 2, 4 . Draw the graph of the open sentence $y = |x|$ within the confines of your coordinate paper.

In Problems 3 and 4

(i) Write the sentence in the y-form.

(ii) Find at least three ordered pairs of numbers which satisfy the equation. (Why do we need no more than two points to graph the line? Another point is desirable as a check.)

(iii) Draw the graph to its full extent on your paper.

3. With reference to one set of axes, draw the graphs of the following.

   (a) $2x - y = 0$ (d) $x + 3y = 0$

   (b) $3x - y = 0$ (e) $x - y = 0$

   (c) $x - 2y = 0$ (f) $x + y = 0$

   What is true about the graphs of all these open sentences?

4. With reference to one set of axes, draw the graphs of the following.

   (a) $3x - 2y = 0$ (d) $3x - 2y = -6$

   (b) $3x - 2y = 6$ (e) $3x - 2y = -12$

   (c) $3x - 2y = 12$

   What is true of the graphs of all these open sentences?

5. Draw the graphs of each of the following with reference to a different set of axes.

   (a) $2x - 7y = 14$ (c) $2x - 7y < 14$

   (b) $2x - 7y > 14$ (d) $2x - 7y \geq 14$

6. With reference to one set of axes draw the graphs of each of the following.

   (a) $5x - 2y = 10$ (c) $5x + y = 10$

   (b) $2x + 5y = 10$ (d) $3x - 4y = 6$

   Which point seems to lie on three of these lines? Do its coordinates satisfy the open sentences associated with these three lines?

7. With reference to one set of axes, draw the graphs of each of the following.

   (a) $2x - 3y = 10$ (c) $3x + 2y = 5$

   (b) $-x + 2y = \frac{1}{2}$ (d) $\frac{1}{2}x - \frac{2}{3}y = 12$

8. Draw the graphs of the open sentences. (Find at least ten ordered pairs satisfying each equation.)

   (a) $y = x^2$ (c) $y = x^2 + 1$

   (b) $y = -x^2$ (d) $y = \frac{1}{x}$

   Are the graphs of these open sentences lines? How do these open sentences differ from those considered in previous problems in this chapter? Can we say that the graph of every open sentence is a straight line?

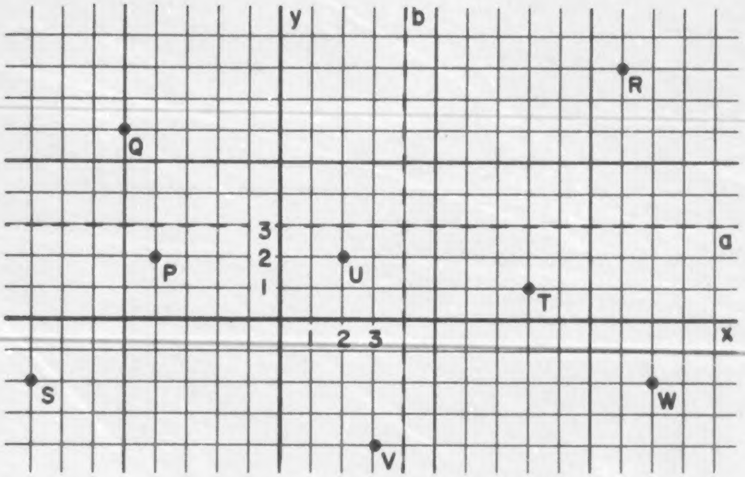

Figure 6.

*9. In Figure 6 are drawn two sets of axes, as indicated: the (x, y)-axes and the (a, b)-axes.

(a) For each of the points P through W, give the coordinates with respect to each set of axes, as is indicated below for P.

Point	(x, y)	(a, b)
P	(-4, 2)	(-8, -1)
Q		

etc.

(b) Give the (a, b) coordinates of the points whose (x, y) coordinates are (5, -5); (-3, -4); (-1, 0); (3, 5) .

14-3. Slopes and Intercepts

With reference to one set of axes, draw the graphs described in (a) through (k) below:

(a) The ordered pairs of numbers for which the ordinate is equal to the abscissa.

Fill in the blanks in the table below so that the pairs satisfy the stated condition:

x	-6		$-2\frac{1}{2}$		3		6
y	-6	-3		0		5.1	6

Now connect successive points with lines. What seems to be true of these lines? Which points given by the table do not lie on the line through (-6, -6) and (6, 6)? Is the point (8, 8) on the line? Extend the line in both directions, as far as possible.

What is the open sentence which describes this graph for all points in the plane? What does this line do to the angles formed by the vertical and horizontal axes?

(b) The ordered pairs of numbers for which the second is the opposite of the first. Fill in the table below:

x	-6		-4.3		2.5		6.1
y		5.1		0		-4	

Could you determine pairs which fulfill the condition without making the table? If you can do this mentally, you will not always need the table. Can you draw a line through all of these points? Extend it as far as possible in both directions.

What is the open sentence which describes this line? How does it differ from the open sentence of the line in (a)?

(c) The ordered pairs for which the ordinate is twice the abscissa. Try to draw this graph without using a table of pairs of numbers.

For this and the ones which follow, make a table if necessary. Draw a line extending as far as possible to include points which fulfill the condition, and write an open sentence which describes the graph.

(d) The ordered pairs for which the ordinate is six times the abscissa.

(e) The ordered pairs for which the ordinate is three times the abscissa.

(f) The ordered pairs for which the ordinate is -6 times the abscissa.

(g) The ordered pairs for which the ordinate is -3 times the abscissa.

(h) The ordered pairs for which the ordinate is one-half the abscissa.

(i) The ordered pairs for which the ordinate is the opposite of one-half the abscissa.

(j) The ordered pairs for which the ordinate is one-sixth of the abscissa.

(k) The ordered pairs for which the ordinate is the opposite of one-fifth of the abscissa.

Problem Set 14-3a

Refer to the eleven graphs just drawn and their corresponding sentences to answer the following questions: (Notice that each of the sentences you have written is in the y-form.)

1. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = x$" and "$x = 0$". What do you observe about these coefficients?

2. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = 0$" and "$y = x$". What is true of these coefficients?

3. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = 0$" and "$y = -x$". What is true of these coefficients?

4. List the coefficients of x in the open sentences for which the lines lie between the graphs of "$y = -x$" and "$x = 0$". What is true of these coefficients?

5. In what position would you expect to find the graph of each of the following open sentences: $y = .01x$, $y = -100x$, $y = -56x$, $y = -\frac{5}{6}x$, $y = \frac{5x}{12}$, $y = \frac{24x}{25}$, $y = -\frac{25x}{24}$?

6. Make a list of information concerning a set of lines containing the origin relating the positions of the lines and the coefficients of x. (Note: a point lies on a line and a line contains a point if it passes through the point.)

7. What can you say about the graphs of equations of the form "$y = kx$", where k is a real number?

   What do you know about the graph of "$y = kx$" when k is positive? When k is negative? When k is between 0 and 1 ? When $k > 1$? When $k < -1$? When $|k| > 1$? When $|k| < 1$? When k is 0 ?

In the preceding problems we have been considering open sentences whose graphs are lines through the origin, and in Problem 7 we saw that the direction of such a line is determined by the coefficient of x. Now let us consider some lines which possibly do not lie on the origin. Graph the following open sentences with reference to the same coordinate axes:

(a) $y = \frac{2}{3}x$ (b) $y = \frac{2}{3}x + 4$ (c) $y = \frac{2}{3}x - 3$

For the first of these no table of values should be necessary. We need simply note that the ordinate must be $\frac{2}{3}$ of the abscissa. In order to get points which are easy to locate we could choose multiples of 3 for values of x. To draw the graph of the second open sentence, we should note that to each ordinate in the graph of the first we add 4. How could we find the ordinates of points for the third open sentence?

What are the coordinates of the points at which lines (a), (b), and (c) intersect the vertical axis? Do you see any relation between these points and equations (a), (b), and (c) ? We call 0, 4, and -3 the *y-intercept numbers* of their respective equations. Points $(0, 0)$, $(0, 4)$ and $(0, -3)$ are the *y-intercepts* of the respective lines. Explain how the graphs of "$y = \frac{2}{3}x + 4$" and "$y = \frac{2}{3}x - 3$" could be obtained by moving the graph of "$y = \frac{2}{3}x$". Notice that the coefficient of x again determines the direction of the lines, whereas the y-intercept numbers determine their positions.

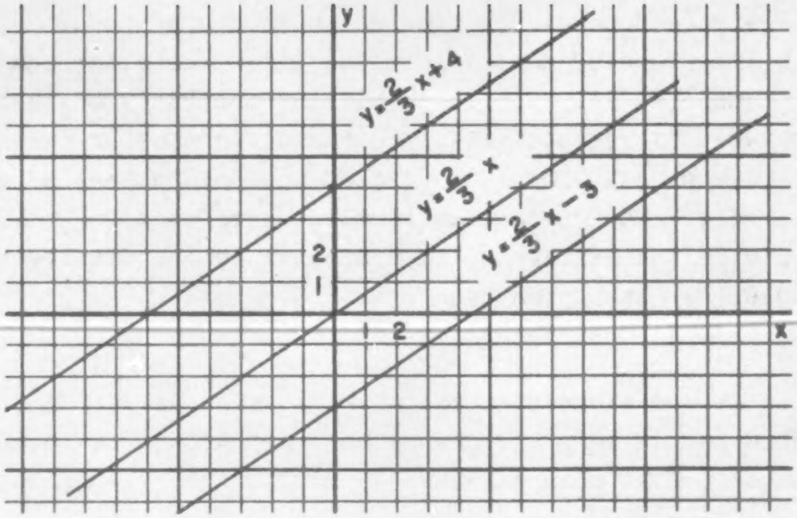

Figure 7.

Write two open sentences such that the absolute value of the y-intercept number is 6 and the coefficient of x is $\frac{2}{3}$. Draw the graphs of these open sentences.

In the figures which you drew in the first part of this section, all of the lines had the same y-intercept, but many different directions. We say that:

> The slope of a line is the coefficient of x in the corresponding sentence written in the y-form. It is the number which determines the direction of the line.

The slope may be either positive, negative or 0. For what positions of the line is the slope negative? 0? What is the slope of the line $x = 2$? Can the equation $x = 2$ be written in y-form? Remember that only non-vertical lines have slopes.

In Figure 8 we have a line which is the graph of "$y = \frac{5}{2}x - 3$". What is the slope of this line? The line passes through points (2, 2) and (4, 7). Verify this. The ordinates of these points are 2 and 7, respectively, and the difference of these ordinates is $7 - 2$, or 5. The abscissas of the points are 2 and 4, respectively. The difference of these abscissas is $4 - 2$, or 2. If we divide the difference in ordinates by the difference in abscissas, we obtain the number

$$\frac{7-2}{4-2} = \frac{5}{2} .$$

But this is the slope of the line! We think of the difference in ordinates as the vertical change and the difference in abscissas as the horizontal change from (2, 2) to (4, 7). Thus,

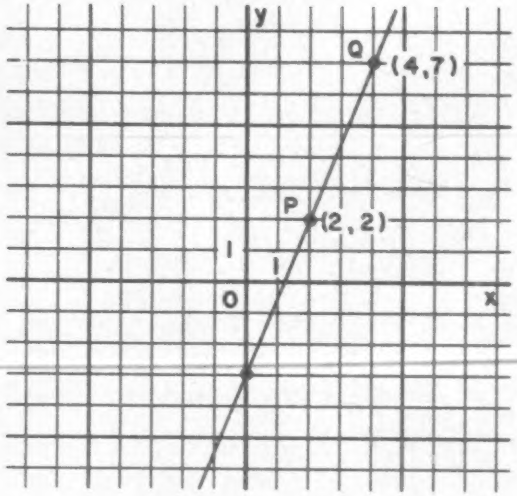

Figure 8.

$$\frac{\text{vertical change}}{\text{horizontal change}} = \frac{7 - 2}{4 - 2} = \frac{5}{2}.$$

Note the order observed in finding these differences: If the first number in the numerator is the ordinate 7 of the point (4, 7), the first number in the denominator must be the abscissa 4 of the same point. What value would we find for the slope if we used as the first number in numerator and denominator the ordinate and abscissa, respectively, of the point (2, 2) ? How would this value compare with the value just found?

We can now prove that the ratio of the vertical change to the horizontal change from one point to another on a line will always be the slope of the line.

Theorem 14-3. Given two points P and Q on a non-vertical line, the ratio of the vertical change to the horizontal change from P to Q is the slope of the line.

*Proof: Consider the non-vertical line whose equation is

$$Ax + By + C = 0,\ B \neq 0 .$$

(Why must we make the restriction $B \neq 0$?) Let us write this in y-form:

$$y = -\frac{A}{B}x + -\frac{C}{B} .$$

Thus, by definition the slope of this line is $-\frac{A}{B}$. Next, consider two points P and Q on the line with coordinates (c, d) and (a, b) respectively. Since these points are on the line, their coordinates satisfy its equation, giving the true sentences

$$Aa + Bb + C = 0$$
$$Ac + Bd + C = 0 .$$

If we subtract the members of these two equations, we have the true sentence

$$A(a - c) + B(b - d) = 0 .$$

This may be written as

$$\frac{b - d}{a - c} = -\frac{A}{B} . \quad \text{(Why?)}$$

But $b - d$ is the vertical change and $a - c$ is the horizontal change from P to Q. This proves the theorem.

What is the slope of the line containing (6, 5) and (-2, -3)? Containing (2, 7) and (7, 3)?

Problem Set 14-3b

1. Find the slope of the line through each of the following pairs of points.

   (a) (-7, -3) and (6, 2)

   (b) (-7, 3) and (8, 3)

(c) (8, 6) and (-4, -1)
(d) (3, -12) and (-8, 10)
(e) (4, 11) and (-1, -2)
(f) (6, 5) and (6, 0)
(g) (0, 0) and (-6, -2)
(h) (0, 0) and (-7, 4)

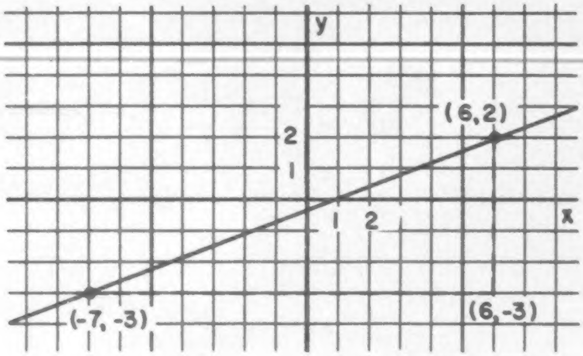

Figure 9.

In Figure 9 we note that the slope of the line is $\frac{2-(-3)}{6-(-7)}$ or $\frac{5}{13}$. We could check this by counting the squares, finding that from (-7, -3) to (6, 2) there are 5 units in the vertical change and 13 units in the horizontal change. It would be possible to write the open sentence of this line with a bit more information, that is, if we knew what the y-intercept number is.

In the case of the line in Figure 10 we note that it contains the points $(-6, 6)$ and $(0, -2)$. From this fact we can determine the slope to be $\frac{6 - (-2)}{(-6) - 0} = -\frac{8}{6} = -\frac{4}{3}$. We know, then, that $y = -\frac{4}{3}x$ is the equation of a line with the same slope as the one we have in Figure 10, but which contains the origin. We see that the y-intercept number of the line in Figure 10 is -2; hence, its equation is "$y = -\frac{4}{3}x - 2$" . What is the equation for a line parallel to the line in Figure 10, but which contains the point $(0, 6)$?

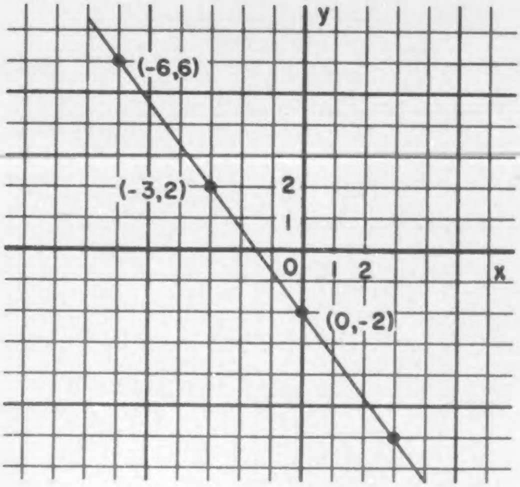

Figure 10.

Problem Set 14-3c

1. What is the equation of a line which contains the point $(0, 6)$ and is parallel to the line whose equation is $y = \frac{4}{3}x - 2$?

2. What is the equation of a line parallel to $y = \frac{4}{3}x - 2$ and containing the point $(0, -12)$?

3. What is the slope of all lines parallel to $y = -\frac{4}{3}x$?

4. What is the slope of all lines parallel to $y = -\frac{2}{3}x$?

5. What is the equation of a line whose slope is $-\frac{5}{6}$ and whose y-intercept number is -3 ?

6. What is the open sentence of a line which passes through $(4, 11)$ and $(2, 4)$ and has y-intercept $(0, -3)$?

7. What is the equation of the line which contains $(5, 6)$ and $(-5, -4)$ and has y-intercept number 0 ?

Now let us see how the slope and the y-intercept can help us to draw lines. Suppose a line has slope $-\frac{2}{3}$ and y-intercept number 6. Let us draw the line as well as write its open sentence. To draw the graph, we start at the y-intercept $(0, 6)$. Then we use the slope to locate other points on the line.

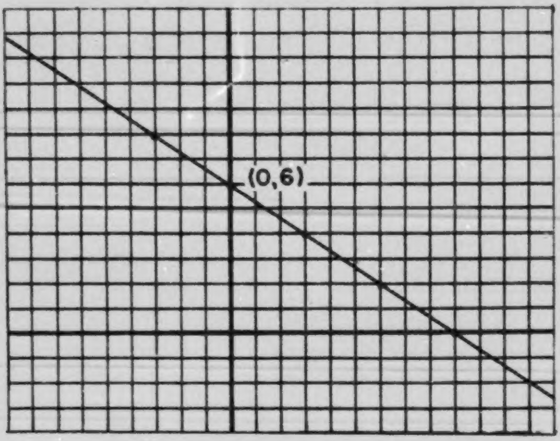

Figure 11.

The fact that the slope is $-\frac{2}{3}$ tells us that between two certain points on the line, the vertical change is -2 while the horizontal change is 3; between two points the vertical change is 4 while the horizontal change is -6; etc. A list of some of the possibilities is

Vertical change	Horizontal change
-2	3
2	-3
4	-6
-10	15

In every case the ratio is $-\frac{2}{3}$. If we take the point which we know is on the line, $(0, 6)$, as one of the two points, we can find another point 3 units to the right and 2 units down; another point is 3 units to the left and 2 units up. We can repeat this process as often as we wish, and quickly get several points through which we may draw the line. Write the open sentence for the line. How would we have chosen the points with respect to $(0, 6)$ if the slope had been $\frac{2}{3}$? What is the open sentence of this line?

What is the form of an equation of a line which has no slope? What is the slope of the line "$x = 2$"? What is an equation of the line through $(-3, 4)$ which has no slope?

Problem Set 14-3d

1. With reference to a set of coordinate axes, select the point $(-6, -3)$ and through this point
   - (a) draw the line whose slope is $\frac{5}{6}$. What is an equation of this line?
   - (b) Draw the line through $(-6, -3)$ which has no slope. What is an equation of this line?
2. Draw the following lines.
   - (a) through the point $(-1, 5)$ with slope $\frac{1}{2}$.
   - (b) through the point $(2, 1)$ with slope $-\frac{1}{2}$.
   - (c) through the point $(3, 4)$ with slope 0.
   - (d) through the point $(-3, 4)$ with slope 2.
   - (e) through the point $(-3, -4)$ with no slope. (What type of line has no slope?)
3. Consider the line containing the points $(1, -1)$ and $(3, 3)$. Is the point $(-3, -9)$ on this line? (Hint: determine the slope of the line containing $(1, -1)$ and $(3, 3)$; then determine the slope of the line containing $(1, -1)$ and $(-3, -9)$.)
4. (a) What do the lines whose open sentences are "$y = x$", "$y = 5x$", "$y = -6x$", "$y = \frac{x}{2}$" have in common?
   - (b) What do the lines whose open sentences are "$y = \frac{1}{2}x - 3$", "$y = \frac{1}{2}x + 4$", "$y = \frac{2x}{4} - 7$" have in common?
   - (c) What do the lines whose open sentences are "$y = \frac{1}{2}x - 3$", "$y = \frac{3x}{4} - 3$", "$y = \frac{7x}{6} - 3$" have in common?

(d) What do the lines whose open sentences are "$x + 2y = 7$", "$\frac{1}{2}x + y = 3$", and "$2x + 4y = 12$" have in common? Show that your answer is correct by drawing the graphs of these three lines.

5. Given the equations.

(a) $3x + 4y = 12$ (b) $2x - 3y = 6$

What is the y-intercept number of each? Draw their graphs. Write each equation in the y-form. What is the slope of each line? Check with its graph.

6. Write each of the following equations in the y-form. Using the slope and the y-intercept, graph each of the lines.

(a) $2x - y = 7$ (c) $4x + 3y = 12$

(b) $3x - 4y = 12$ (d) $3x - 6y = 12$

Are you certain that the graphs of these open sentences are lines? Why?

7. Write an equation of each of the following lines.

(a) The slope is $\frac{2}{3}$ and the y-intercept number is 0.

(b) The slope is $\frac{3}{4}$ and the y-intercept number is -2.

(c) The slope is -2 and the y-intercept number is $\frac{4}{3}$.

(d) The slope is -7 and the y-intercept number is -5.

(e) The slope is m and the y-intercept number is b. Can the equation of every straight line be put in this form? What about the equations of the coordinate axes?

8. Given points $(0, 0)$ and $(3, 4)$ with a line containing them. What is the slope of the line? What is its y-intercept? Write the equation of the line.

9. Write the equation of the line whose y-intercept number is 7 and which contains the point $(6, 8)$. What is the slope of the line? Can you write the slope as $\frac{8 - 7}{6 - 0}$?

10. What is the slope of the line which contains $(-3, 2)$ and $(3, -4)$? If (x, y) is a point on this same line, verify that the slope is also $\frac{y - 2}{x - (-3)}$. Also verify that $\frac{y - (-4)}{x - 3}$ is the slope. If -1 and $\frac{y - 2}{x - (-3)}$ are different names for the slope, show that the equation of the line is "$y - 2 = (-1)(x + 3)$". Show that it can also be written "$y + 4 = (-1)(x - 3)$" .

11. Write the equations of the lines through the following pairs of points. (Try to use the method of Problem 10 for parts (e) - (h).)

(a) $(0, 3)$ and $(-5, 2)$

(b) $(5, 8)$ and $(0, -4)$

(c) $(0, -2)$ and $(-3, -7)$

(d) $(5, -2)$ and $(0, 6)$

(e) $(-3, 3)$ and $(6, 0)$

(f) $(-3, 3)$ and $(-5, 3)$

(g) $(-3, 3)$ and $(-3, 5)$

(h) $(4, 2)$ and $(-3, 1)$

*12. Any polynomial of first degree in one variable x of the form "$kx + n$", where k and n are real numbers, is said to be linear in x. It is called linear, since the graph of the open sentence "$y = kx + n$" is a straight line. The graph of "$y = kx + n$" is also called the graph of the polynomial "$kx + n$". Draw the graph of each of the following linear polynomials:

(a) $2x - 5$

(b) $-2x + \frac{1}{2}$

(c) $\frac{2}{3}x - 1$

(d) $-\frac{3}{2}x + 2$

*13. Consider a rectangle whose length is 3 units greater than its width w .

(a) Write an expression in w whose value for each value of w is equal to the perimeter of the rectangle. Is this a linear expression in w ?

(b) Write an expression in w for the area of the rectangle. Is this a linear expression in w ?

*14. Consider a circle of diameter d .

(a) Write an expression in d for the circumference of the circle. Is this expression linear in d ? What happens to the circumference if the diameter is doubled? Halved? If c is the circumference, what can you say about the ratio $\frac{c}{d}$? How does the value of $\frac{c}{d}$ change when the value of d is changed?

(b) Write an expression in d for the area of the circle. Is this expression linear in d ? Is it linear in d^2 ? If A is the area of the circle, what can you say about the ratio $\frac{A}{d}$? What about the ratio $\frac{A}{d^2}$? Does the value of the ratio $\frac{A}{d}$ change when the value of d is changed? Does the value of $\frac{A}{d^2}$ change when d is changed?

*15. In the case of the special linear expression "kx", the value of the expression is said to vary directly as the value of the variable x. The coefficient k is called the constant of variation. The value of the expression kx^2 is said to vary directly as the square of the value of x.

(a) Does the circumference of a circle vary directly as the diameter? What is the constant of variation in this case? Does the area of the circle vary directly as the diameter? Does the area vary directly as the square of the diameter? What is the constant of variation?

(b) In terms of a graph, what does the constant of variation mean if y varies directly as x ?

(c) If the constant of variation is negative, what can you say as to the way in which the value of the expression varies when you change the value of the variable?

(d) What would be the form of an expression in one variable x such that the value of the expression varies directly as the square root of x ?

*16. An automobile is moving at a constant speed along a straight road. If t is the time in hours since the start, write an expression in t whose value is the distance traveled in miles. Is this expression linear in t ? Does the distance vary directly as the time? How can you interpret the constant of variation in this example? If it is known that the automobile has traveled 25 miles at the end of 20 minutes, what is the constant of variation?

*17. In the case of an expression of the form $\frac{k}{x}$, the value of the expression is said to vary inversely as the value of x. The number k is the constant of variation.

(a) Draw the graphs of the open sentences:

$$y = \frac{1}{x}; \quad y = -\frac{1}{x}; \quad y = \frac{2}{x}.$$

(b) If the variable x is given increasing positive values, what can you say of the values of $\frac{k}{x}$? Do they increase or decrease? Does it matter whether k is positive or negative?

*18. A rectangle has an area of 25 square units, and one side has length w units.

(a) Write an expression in w for the length of the other side.

(b) Is this a case of inverse variation? What is the constant of variation?

(c) Draw the graph of the expression in (a).

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*14-4. Graphs of Open Sentences Involving Integers Only

In drawing graphs of open sentences, we must keep in mind that every point of a graph is associated with some pair of real numbers. Suppose we consider a sentence in which the values of the variables are restricted to integers, so that the coordinates of points on the graph must be integers. What would such a graph look like?

First let us consider the coordinate axes. Would they still be straight lines? It seems that they are sets of points such as (0, 1), (0, 2), (0, 3), etc., since we are restricting ourselves to integers, and we might wish to distinguish the axes for such cases from the coordinate axes for all real numbers. However, a series of dots would be apt to be confused with the graph itself; so we use a series of short dashes for each axis.

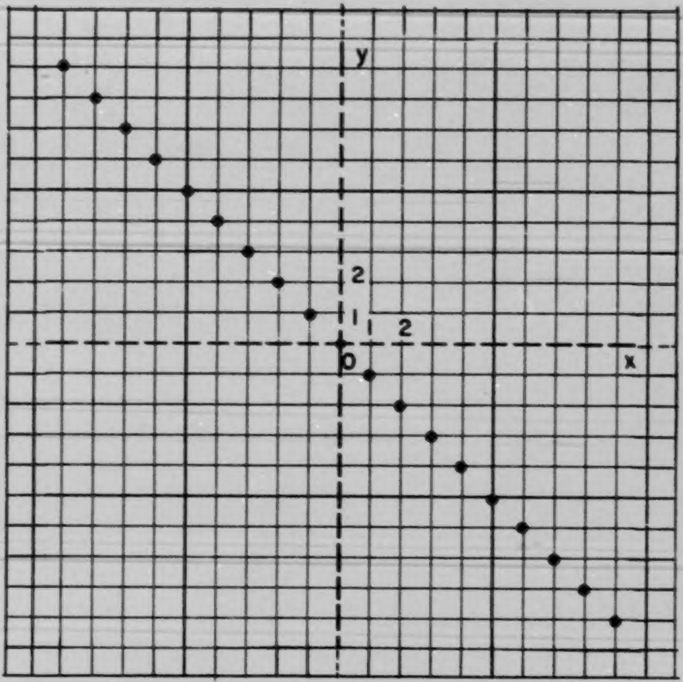


Figure 12.

What is the open sentence associated with the graph in Figure 12? In determining this, we first note that the graph includes points with integral coordinates only, and second that each ordinate is the opposite of the corresponding abscissa. This may be stated as follows: "$y = -x$, where x and y are integers such that $-10 < x < 10$ and $-10 < y < 10$".

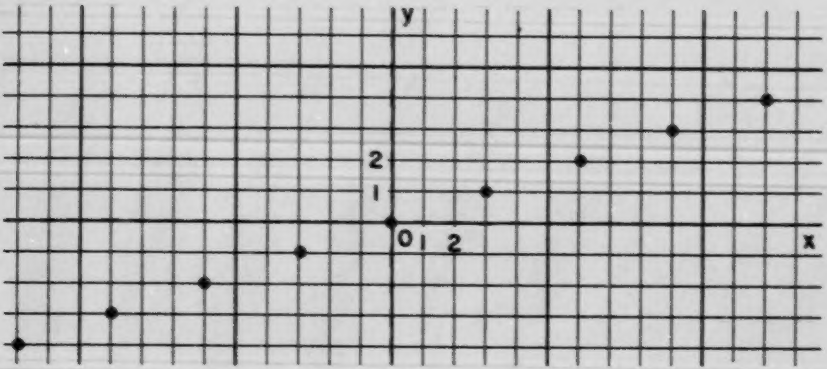


Figure 13.

In Figure 13 the points go on beyond the limit of this diagram. Note that there are no points for $x = 1$, $x = 2$, $x = 4$, $x = -1$, and others. What do you notice about the ordinate corresponding to each abscissa, if we assume that all the points lie on a straight line, as these points seem to indicate? We would write the open sentence: "$y = \frac{x}{3}$ where x and y are integers". Why can the abscissa not be 1 or 2 ?

Consider Figure 14. For this set of twelve points it seems there is no simple open sentence. Can you describe the limitations on the abscissas? What statement can you make about the ordinates?

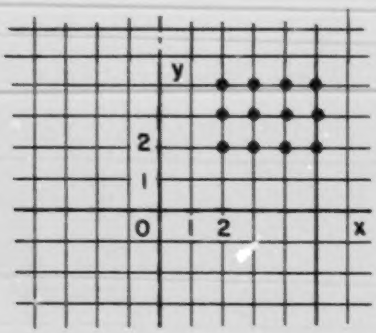


Figure 14.

These facts could be stated in a compound open sentence as follows: "$1 < x < 6$ and $1 < y < 5$, where x and y are integers" .

Notice that here the connective for the compound sentence is and; note also that the points whose coordinates make the sentence true are only those which belong to the truth sets of both parts of the compound sentence.

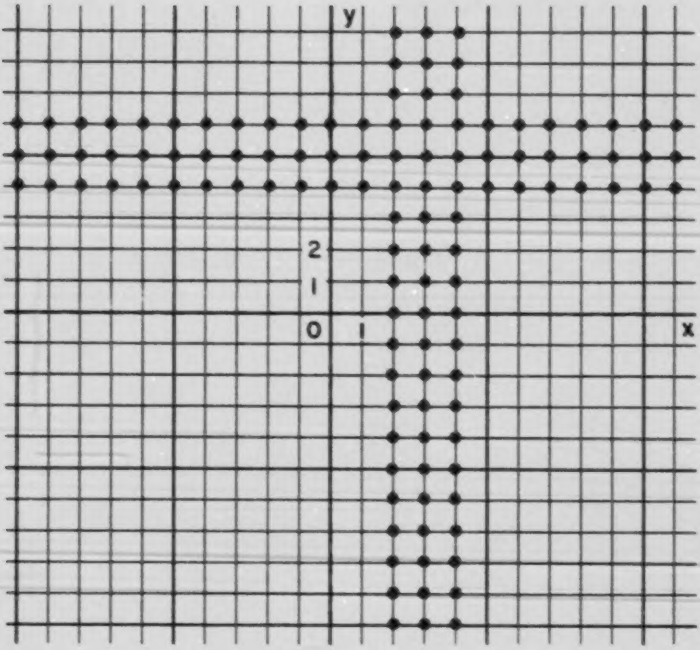


Figure 15.

In Figure 15 a different situation exists. Let us see what open sentence will describe this graph. The three horizontal rows of dots could be the graph of the sentence: "$3 < y < 7$, where x and y are integers". Then we write a sentence which describes the three vertical rows of dots: "$1 < x < 5$ where x and y are integers". The open sentence which describes the total set of points is "$1 < x < 5$ or $3 < y < 7$, where x and y are integers". Another way of stating this would be: "$2 \leq x \leq 4$ or $4 \leq y \leq 6$ where x and y are integers". Notice that the connective here is or, and that the graph includes all points which belong to the truth sets of either of the two parts of the compound sentences, or to both of them.

Problem Set 14-4

1. With reference to separate sets of coordinate axes, and for x and y integers, draw the graph of each of the following.

   (a) $y = \frac{x}{2}$, for $-6 < x < 6$

   (b) $y = 3x - 2$

   (c) $y = 2x + 4$

2. Draw the graphs of each of the following with reference to a separate set of coordinate axes.

   (a) $-3 < x < 2$ and $-2 < y < 1$, where x and y are integers.

   (b) $-3 < x < 2$ or $-2 < y < 1$, where x and y are integers.

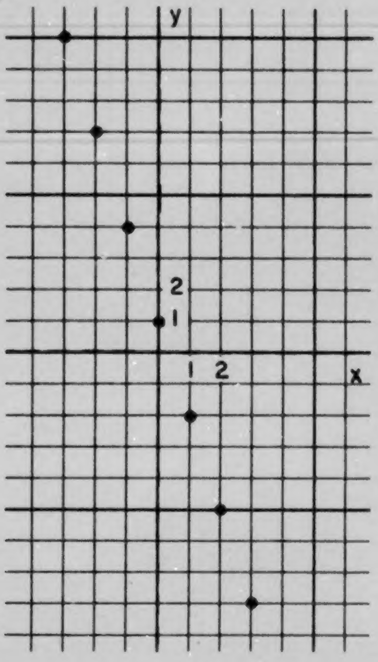
   (c) $5 \leq x \leq 6$ or $1 \leq y \leq 3$, where x and y are integers.

   (d) $5 \leq x \leq 6$ and $y = 0$, where x and y are integers.

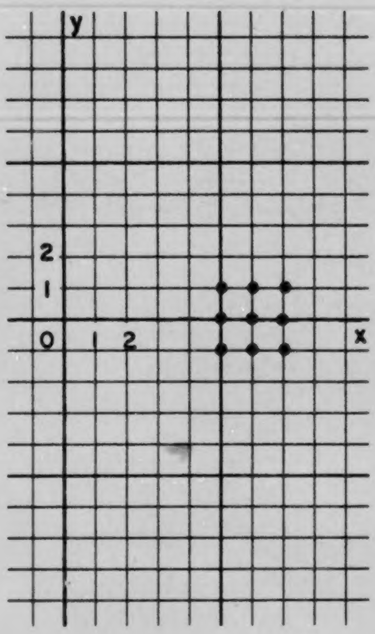
3. Write a compound open sentence whose truth set is $\{(-1, 3)\}$.

4. Write open sentences whose truth sets are the following sets of points:

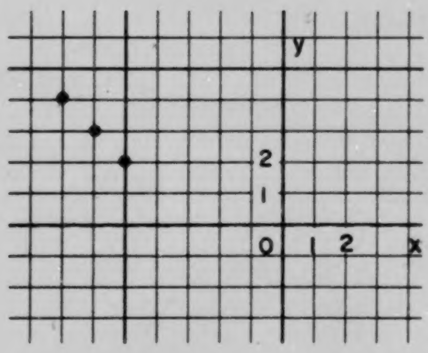
(a)



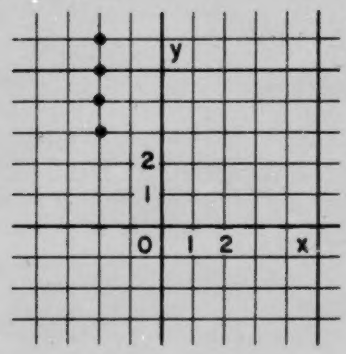
(b)



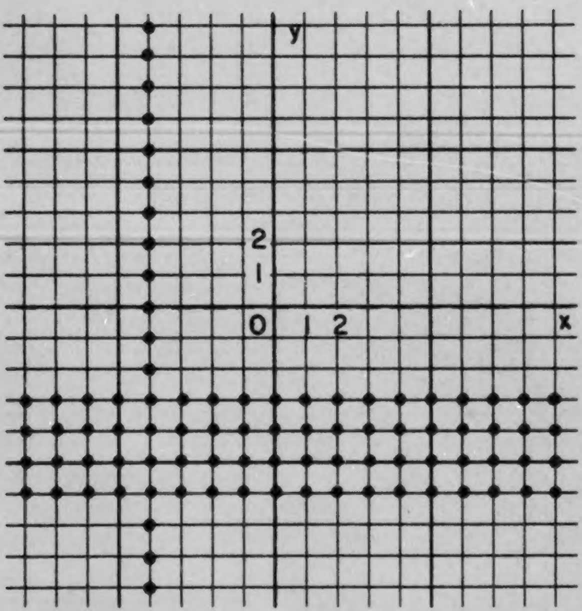
(c)



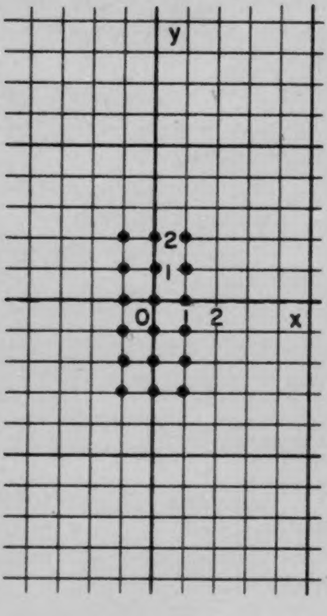
(d)



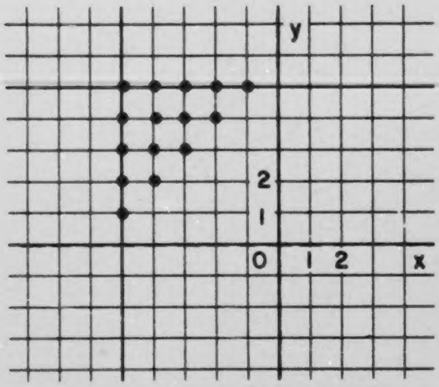
(e)



(f)



(g)



5. With reference to separate sets of coordinate axes, draw the graphs of the following.

   (a) $x + 5y = 15$, where x and y are integers.

   (b) $x + 5y = 15$, where x and y are real numbers. How do the graphs of (a) and (b) differ? Name some points on one graph which are not on the other.

   (c) $x + 5y = 15$, where x and y are rational numbers. How does this differ from the other graphs? If (x, y) is a point on the line $x + 5y = 15$ and if x is rational, what about y ?

14-5. Graphs of Open Sentences Involving Absolute Value

Consider the equation "$|x| = 3$". This sentence is equivalent to the sentence "$x = 3$ or $-x = 3$". What would its graph look like? First consider the graph for "$x = 3$". The graph of this open sentence is a straight line parallel to the vertical axis and three units to the right of it. The graph of "$-x = 3$" is a second line parallel to the vertical axis and three units to the left of it. Hence, the complete graph of $|x| = 3$ consists of two lines which are the graphs of "$x = 3$", "$-x = 3$", as in Figure 16. Describe and draw the graphs of:

(a) $|x| = 5$ (b) $|x| = 7$ (c) $|y| = 2$ (d) $|y| = 3$

For what value of k will the graph of $|x| = k$ be a single line?

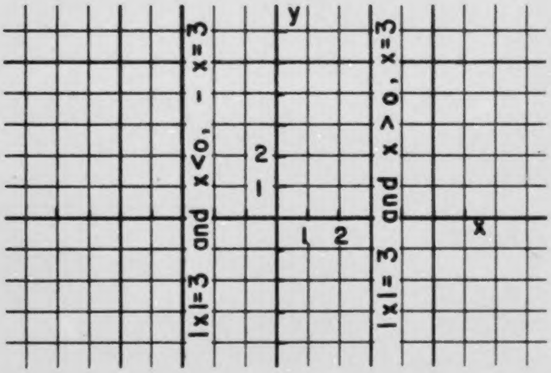


Figure 16.

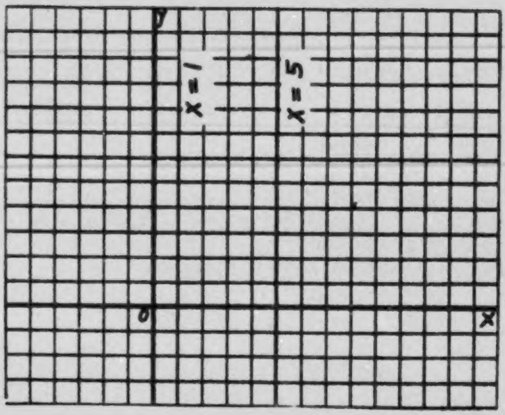


Figure 17.

Consider the graph of the open sentence "$|x - 3| = 2$".

This sentence is equivalent to "$x - 3 = 2$ or $-(x - 3) = 2$". What would its graph look like? First consider the open sentence $x - 3 = 2$, whose graph is a straight line parallel to the vertical axis and five units to the right of it. Then note that the graph of "$-(x - 3) = 2$", that is, of "$x = 1$", is a line parallel to the vertical axis and 1 unit to the right of it. Hence the complete graph of "$|x - 3| = 2$" consists of two lines, one the graph of "$x = 5$" and the other the graph of "$x = 1$", as in Figure 17. Check on the drawing that one of these lines is two units to the right of $x = 3$, and the other is two units to the left of $x = 3$. Recall that on the number line, "$|x - 3| = 2$" meant that the distance between x and 3 is 2.

Problem Set 14-5a

1. Draw the graph of each of the following open sentences with reference to a different set of axes.

   (a) $|x - 4| = 2$ (d) $|x + 1| = 2$

   (b) $|y - 2| = 3$ (e) $|x + 3| = 1$

   (c) $|y| = 5$ (f) $|y + 2| = 3$

2. Draw the graph of each of the following open sentences with reference to a different set of axes.

   (a) $|x| > 2$

   (b) $|x| \geq 2$

   (c) $|x| < 5$

3. Draw the graph of each of the following with references to a different set of axes.

   (a) $y > 2x + 4$ (c) $y \geq \frac{3x}{4} - 1$

   (b) $y < \frac{2x}{3} + 7$ (d) $y \leq 2x - 1$

4. Draw the graph of each of the following with reference to a different set of axes.

   (a) $2x + y > 3$ (c) $x - 2y \leq 4$

   (b) $x + 2y \geq 4$ (d) $2x - y \leq 3$

5. Write the open sentences for which lines (1), (2), (3), and (4) in the figure are the graphs. Notice that (4) actually is a pair of lines.

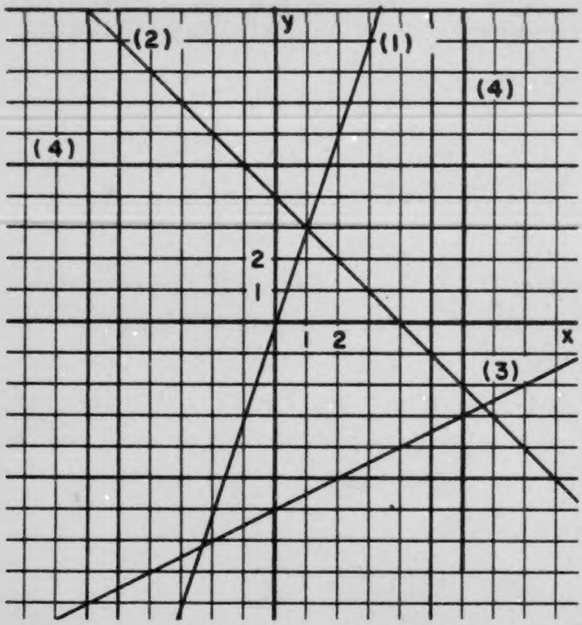


Figure for Problem 5.

6. Draw a set of coordinate axes. With reference to these axes, locate three points from each of the sets described below. For each set draw a line through the three points.

   (a) The second coordinate of the ordered pair is twice the first.

   (b) The second coordinate of the ordered pair is 5 more than one-half the first.

   (c) The second coordinate of the ordered pair is one-half the first.

   (d) The second coordinate of the ordered pair is the opposite of the first.

Let us consider the open sentence "$y = |x|$". Whether x is positive or negative, what is true of the absolute value of x ? What, then, must be true of y for every value of x except 0 ? What is the value of y for $x = 0$?

x	-3	-2	-1	0	1	2	3
$\lvert x\rvert$	3	2	1	0	1	2	3

From Figure 18 we notice something new to us: the graph of the simple sentence "$y = |x|$" turns out to be the two sides of a right angle. Can you guess why this is a right angle? Is it possible to have a simple equation whose graph would be two lines which do not form a right angle? Suggest one.

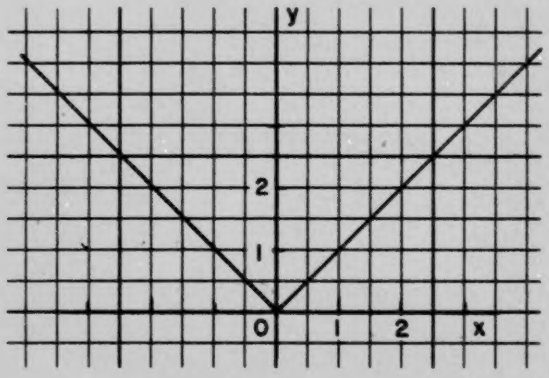


Figure 18.

Problem Set 14-5b

1. Draw the graph of each of the following with reference to a separate set of axes.

   (a) $y = 2|x|$ (d) $y = -2|x|$

   (b) $y = \frac{1}{2}|x|$ (e) $x = -|y|$

   (c) $y = -|x|$ (f) $x = |-2y|$

2. Draw the graph of each of the following with references to a separate set of axes.

   (a) $y = |x| + 3$ (d) $x = |y| + 3$

   (b) $y = |x| - 7$ (e) $x = 2|y| - 1$

   (c) $y = 2|x| + 1$ (f) $y = -|x| - 1$

3. Draw the graph of each of the following with reference to a separate set of axes.

   (a) $y = |x - 2|$ (d) $y = |x + 3| - 5$

   (b) $y = |x + 3|$ (e) $y = \frac{1}{2}|x - 1| + 3$

   (c) $y = 2|x + 3|$

4. How would you get each of the graphs in Problems 1 (c), (e), 2 (a), (b), (d), (f), 3(a), (b), (d) from the graph of either $y = |x|$ or $x = |y|$ by rotating or sliding the graph? Examples: The graph of "$y = |x - 2|$" can be obtained by sliding the graph of "$y = |x|$" to the right 2 units. The graph of "$x = -|y|$" can be obtained by rotating the graph of "$x = |y|$" about the y-axis. The graph of "$y = |x| - 7$" can be obtained by sliding the graph of "$y = |x|$" down 7 units.

*5. What does the graph of $|x| + |y| = 5$ look like? Let us make a chart first. Suppose we start with the intercepts. Let $y = 0$ and get some possible values of x which will make the sentence true. Then let $x = 0$, and get some values of y. Now fill in some of the other possible values.

Suppose $x = 6$, what can you say about possible values for y? If $x = 3$, then $|x| = 3$, and $|y| = 2$; what possible values may y have? Fill in the blanks in the table below, and then draw the graph. How would you describe the figure?

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
$\|x\|$	5	3	3	1		0	0					5
$\|y\|$	0		2	4		5	5					0
y	0		-2	4		5	-5					0

We could write four open sentences from which we could get the same graph, provided we limited the values of x:

$$x + y = 5, \text{ and } 0 \le x \le 5,$$
$$x - y = 5, \text{ and } 0 \le x \le 5,$$
$$-x + y = 5, \text{ and } -5 \le x \le 0,$$
$$-x - y = 5, \text{ and } -5 \le x \le 0.$$

With reference to one set of axes, draw the graphs of the four open sentences stated above. Why was it necessary to limit the values of x?

*6. Draw the graph of each of the following with reference to a separate set of axes.

(a) $|x| + |y| > 5$ (c) $|x| + |y| \le 5$

(b) $|x| + |y| < 5$ (d) $|x| + |y| \nless 5$

*7. Make a chart of some values which make the open sentence

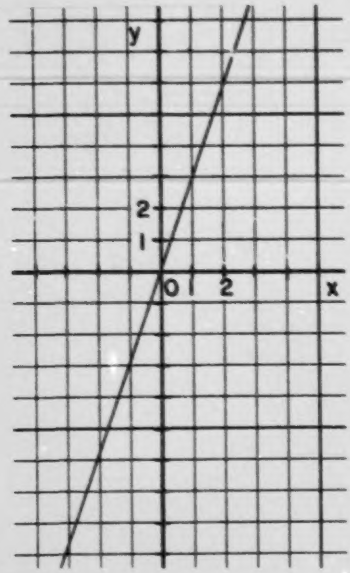
$$|x| - |y| = 3$$

true, and draw the graph of the open sentence. Write four open sentences, as in Problem 5, whose graphs form the same figure.

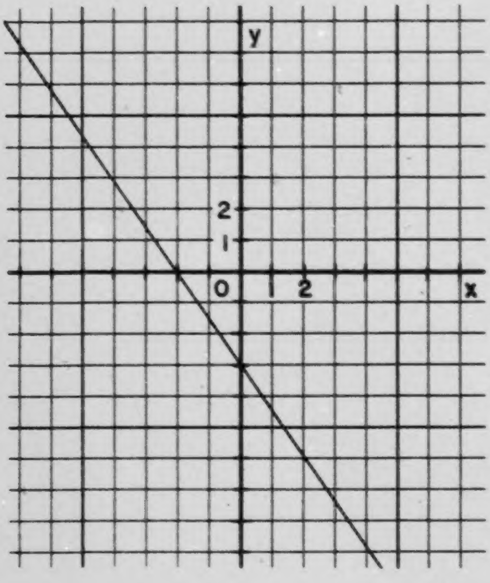
Review Problems

1. For each of the following graphs, write its open sentence.

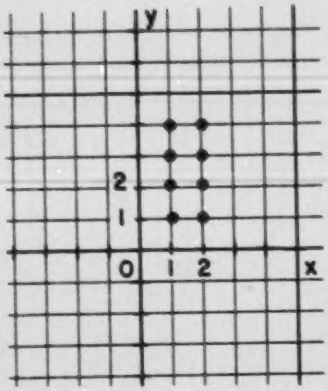
(a)



(b)



*(c)



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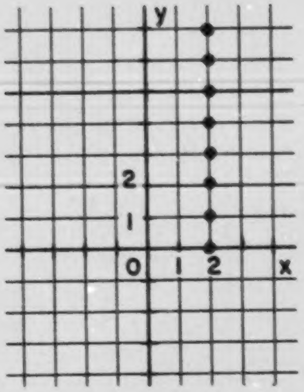
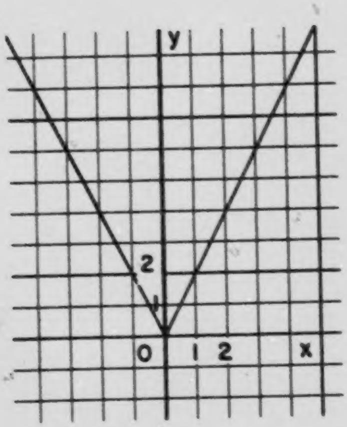
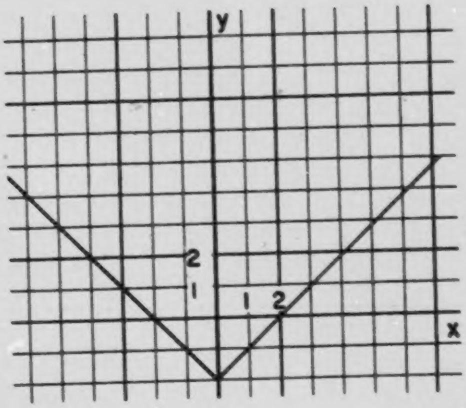


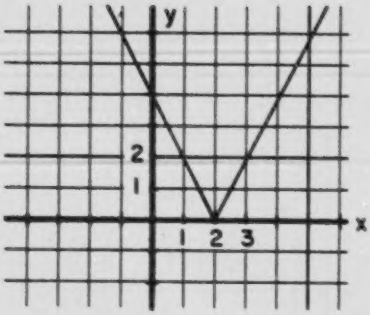
(e)



(f)



(g)



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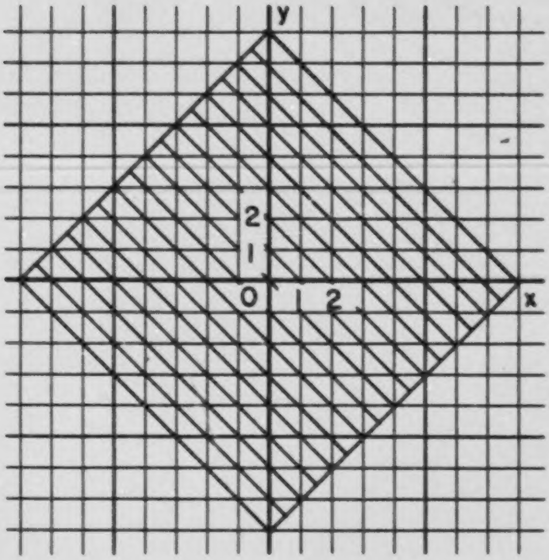
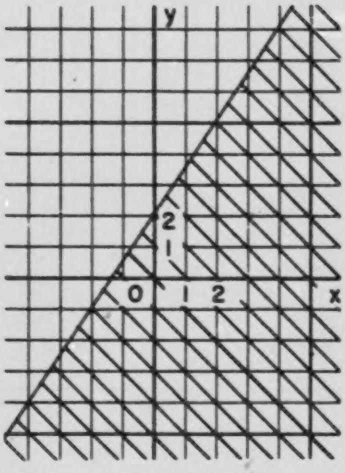
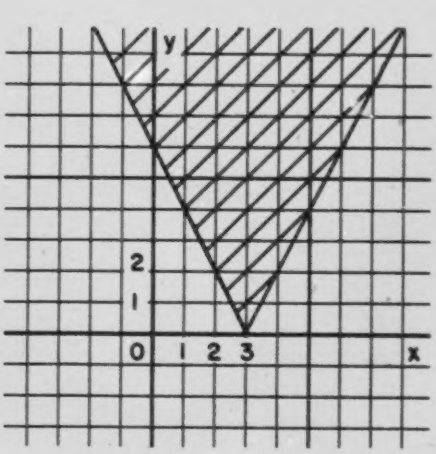


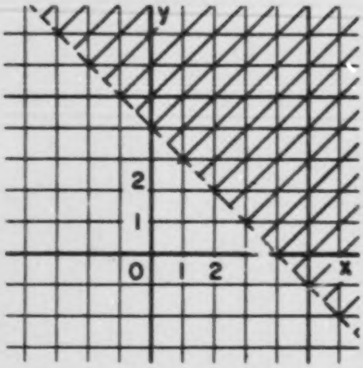
(i)



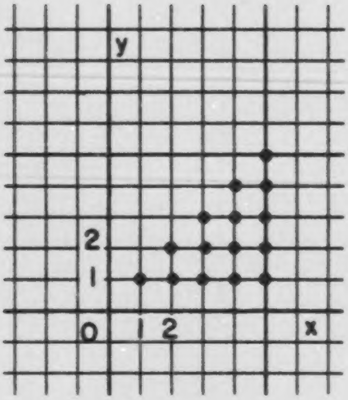
(j)



(k)



*(ℓ)



2. Draw the graph of each of the following open sentences.

(a)	$y \leq 3x$	(g)	$\lvert x\rvert + \lvert y\rvert = -2$
(b)	$y = \frac{x}{2} + 7$	(h)	$3y \geq 2x - 1$
(c)	$y < \frac{x}{2} - 5$	(i)	$x = 3$ and $y = -1$
(d)	$y > 3$	(j)	$x + y \leq -2$
(e)	$x < 1.5$	(k)	$3y - 12 = 0$
(f)	$x + y = 0$		

*3. Draw a set of coordinate axes, designating them as the (x, y)-axes. Through point $(2, -1)$ draw a pair of (a, b)-axes, making the a-axis parallel to the x-axis and the b-axis parallel to the y-axis. Locate the following points with reference to the (x, y)-axes: $A(3, -5)$, $B(-5, 3)$, $C(-2, -5)$, $D(0, 3)$, $E(0, -3)$, $F(-5, -1)$, $G(-4, 3)$, $H(6, 0)$, $I(-6, 0)$, $J(2, 6)$. Make a table giving the coordinates of each of these points with reference to the (a, b)-axes.

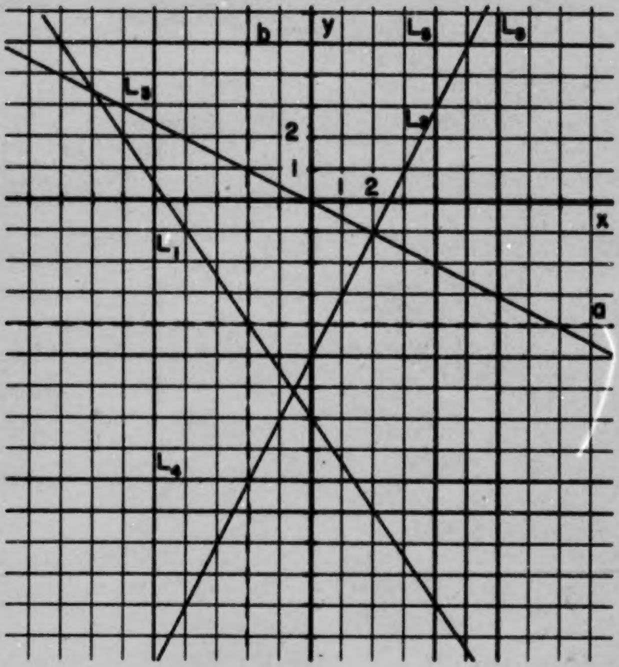

b
y
L5
L5
L3
L2
2
1
1 2
x
L1
a
L4

Figure for Problem 4.

*4. Give two equations for each of the lines in the above figure, one with reference to the (x, y)-axes, the other with reference to the (a, b)-axes. (Note that L_5 is a pair of lines.)

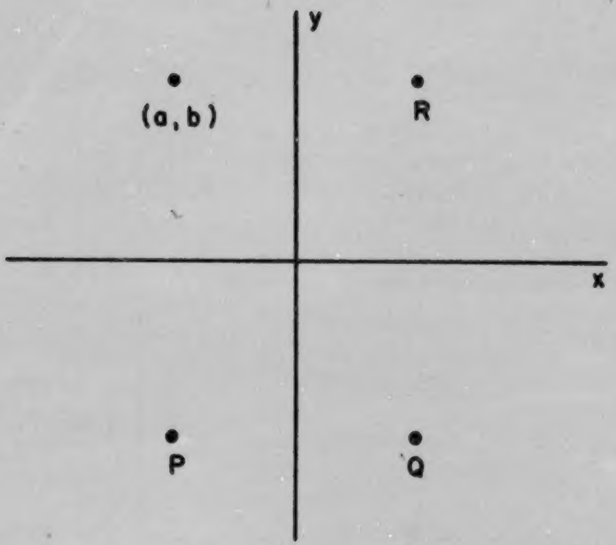


Figure for Problem 5.

5. If the point (a, b) is in the second quadrant

   (a) is a positive?

   (b) is b positive?

   (c) If the coordinates of P, Q, and R have the same absolute values as the abscissa and ordinate of (a, b), state the coordinates of P, Q, and R in terms of a and b.

6. If (c, d) is a point in the third quadrant, in which quadrant is the point (c, -d)? The point (-c, d)? The point (-c, -d)?

*7. Draw the graph of "$y = 3x + 4$". What happens to this graph when its equation is changed as follows?

(a) $y = 3(-x) + 4$	(c) $y = (3x + 4) - 3$
(b) $y = -(3x + 4)$	(d) $y = 3(x - 2) + 4$

*8. Draw the graph of "$y = 2|x|$". Give an equation of the graph which results from each of the following changes.

(a) The graph is rotated one-half revolution about the x-axis.

(b) The graph is moved 3 units to the right.

(c) The graph is moved 2 units to the left.

(d) The graph is moved 5 units up.

(e) The graph is moved 2 units to the right and 4 units down.

9. (a) With reference to one set of axes, draw the graphs of:

$$2x + y - 5 = 0$$

$$6x + 3y - 15 = 0$$

What is true about these two graphs? Now look at the equations; how could you get the second equation from the first?

(b) What is true of the graphs of

$$Ax + By + C = 0$$

and

$$kAx + kBy + kC = 0$$

for any non-zero k ?

(c) Under what condition will the graphs of

$$Ax + By + C = 0$$

and

$$Dx + Ey + F = 0$$

be the same line? If the graphs are the same line, what is true of the ratios

$$\frac{A}{D}, \frac{B}{E}, \text{ and } \frac{C}{F} ?$$

10. (a) With reference to one set of axes, draw the graphs of

$$3x - 4y - 12 = 0,$$
$$12x - 16y - 32 = 0.$$

What is true about these two graphs? What is true about the coefficients of x and y in these equations?

(b) What is true of the graphs of

$$Ax + By + C = 0$$

and

$$kAx + kBy + D = 0$$

for any non-zero k ?

(c) Under what conditions will the graphs of

$$Ax + By + C = 0$$
$$Dx + Ey + F = 0$$

be parallel lines? If the graphs are parallel lines, what equal ratios are there among the coefficients?

11. Draw the graphs of the equations

(a) $7u + 3t - 4 = 0$

(b) $2s - 5v - 1 = 0$

Do the graphs depend on the choices you made for the first variable? Explain why we speak of sentences with two ordered variables.

*12. Suppose a point with coordinates (a, b) is moved to the point $(a, -b)$. Describe this in terms of opposites. Describe it in terms of a rotation. Answer the following questions, and locate the points referred to in parts (a) and (b).

(a) What points do the following points go to?

$(2, 1)$, $(2, -1)$, $(-\frac{1}{2}, 2)$, $(-2, -3)$, $(3, 0)$, $(-5, 0)$, $(0, 5)$, $(0, -5)$.

(b) What points go to the points listed in (a) ?

(c) What point does (a, -b) go to?

(d) What point does (-a, b) go to?

(e) What point goes to (a, b)?

(f) What points go to themselves?

*13. Suppose a point with coordinates (a, b) is moved to the point (a - 3, b + 2). How can you obtain this by moving all the points of the plane? Answer the following questions and locate the points:

(a) What points do the following points go to? (1, 1), (-1, -1), (-2, 2), (0, -3), (3, 0)?

(b) What points go to the above points?

(c) What point does (a, b - 2) go to?

(d) What point goes to (-a, -b)?

(e) Which points go to themselves?

(f) Describe how the points are moved if (a, b) is moved to (a, b - 2).

Chapter 15

SYSTEMS OF EQUATIONS AND INEQUALITIES

15-1. Systems of Equations

We began a study of compound sentences in Chapter 3. What connectives are used in compound sentences? Let us first consider a compound sentence of two clauses in two variables, whose connective is "or" ; for example,

$$x + 2y - 5 = 0 \quad \text{or} \quad 2x + y - 1 = 0 \ .$$

When is a compound sentence with the connective "or" true? The truth set of this sentence includes all the ordered pairs of numbers which satisfy "$x + 2y - 5 = 0$", as well as all the ordered pairs which satisfy "$2x + y - 1 = 0$", and the graph of its truth set is the pair of lines drawn in Figure 1.

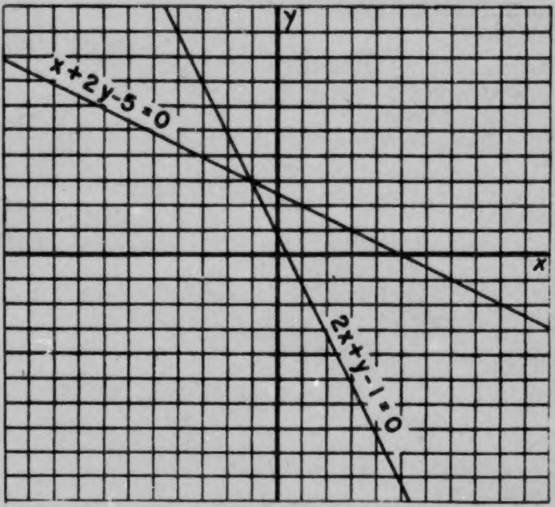


Figure 1.

Name three ordered pairs of numbers which belong to the truth set of

$$x + 2y - 5 = 0 .$$

Name four ordered pairs which belong to the truth set of

$$2x + y - 1 = 0 .$$

Which of these ordered pairs of numbers are elements of the truth set of the compound open sentence

$$x + 2y - 5 = 0 \quad \text{or} \quad 2x + y - 1 = 0 \ ?$$

If we remember that the sentences "$ab = 0$" and "$a = 0$ or $b = 0$" are equivalent when a and b are real numbers, another way of writing this compound sentence would be

$$(x + 2y - 5)(2x + y - 1) = 0 .$$

Now consider a compound sentence with the connective "and" instead of "or". Which ordered pairs are elements of the truth set of the compound open sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$" ? Note that only one ordered pair $(-1, 3)$, satisfies both clauses of this sentence, and therefore the graph of the truth set of the open sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$" is the intersection of the pair of lines in Figure 1.

In this chapter we shall devote most of our attention to compound open sentences made up of two clauses connected by and. This sort of compound open sentence, with the connective "and" , is often written

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 . \end{cases}$$

This is called a system of equations in two variables. When we talk about the truth set of a system of equations we mean the set of elements common to both the truth sets. As we have seen, the truth set of

$$\begin{cases} 2x + y - 1 = 0 \\ x + 2y - 5 = 0 \end{cases}$$

is $\{(-1, 3)\}$.

Problem Set 15-1a

1. Find the truth sets of the following systems of equations by drawing the graphs of each pair of open sentences and guessing the coordinates of the intersections. (In each case, verify that your guess satisfies the sentence.)

(a) $\begin{cases} 2x - y - 2 = 0 \\ 3x + y - 3 = 0 \end{cases}$ (c) $\begin{cases} x - 3 = 0 \\ 2x + 3y - 9 = 0 \end{cases}$

(b) $\begin{cases} 2x + 3y - 3 = 0 \\ x + y = 0 \end{cases}$ (d) $\begin{cases} 5x + y - 10 = 0 \\ 2x + 2y - 8 = 0 \end{cases}$

(e) $\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$

2. Draw the graphs of the truth sets of the following sentences.

(a) $x + 2y - 6 = 0$ or $2x + y - 5 = 0$

(b) $(2x - 3y + 9)(3x + y - 2) = 0$

Did you have trouble guessing the coordinates of the intersection points in problems 1 (d) and (e)? Let us see if we can find a systematic way to obtain the ordered pair without guessing.

Returning to the compound sentence "$x + 2y - 5 = 0$ and $2x + y - 1 = 0$", and looking at Figure 2, we see that there are many compound open sentences whose truth set is $\{(-1, 3)\}$; for example "$2x + y - 1 = 0$ and $y - 3 = 0$", and "$x + 2y - 5 = 0$ and $x + 1 = 0$" are two such equivalent compound sentences, because their graphs are pairs of lines intersecting in $(-1, 3)$. State at least two more compound sentences whose truth set is $\{(-1, 3)\}$. What is the truth set of

$$x + 1 = 0 \text{ and } y - 3 = 0 \text{ ?}$$

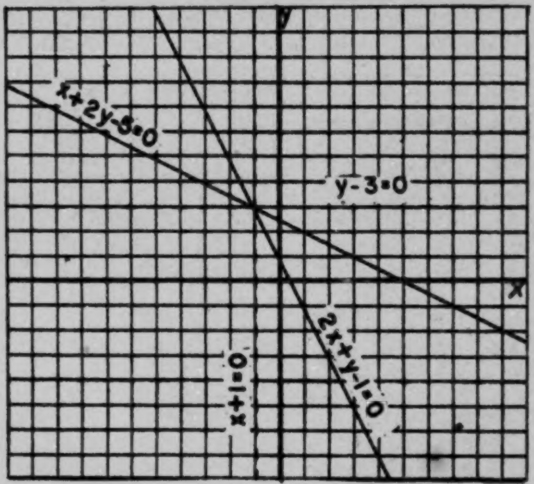


Figure 2.

From this it appears that we could easily find the truth set of any compound open sentence of the type

$$2x + y - 1 = 0 \quad \text{and} \quad x + 2y - 5 = 0$$

if we had a method for getting the equations of the horizontal and vertical lines through the intersection of the graphs of the two clauses.

There are many lines through any point. Here is a method which, as we shall see, will give us the equation of any line through the intersection of two given lines, provided that the lines do intersect in a single point. We shall again use the system

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0 \end{cases}$$

to illustrate.

We multiply the expression on the left of the first equation by any number, say 3, and the expression on the left of the second equation by any number, say 7, and form the sentence

$$3(x + 2y - 5) + 7(2x + y - 1) = 0$$

We see that:

(1) The coordinates of the point of intersection $(-1, 3)$ of the two lines satisfy this new sentence:

$$3\left(-1 + 2(3) - 5\right) + 7\left(2(-1) + 3 - 1\right) = 3(0) + 7(0) = 0$$

In general, we know that a point belongs to the graph of a sentence if its coordinates satisfy the sentence. So the graph of our new open sentence

"$3(x + 2y - 5) + 7(2x + y - 1) = 0$"

contains the point of intersection of the two lines

"$x + 2y - 5 = 0$" and "$2x + y - 1 = 0$".

(2) The graph of the new sentence is a line, because:

$$\begin{aligned} 3(x + 2y - 5) + 7(2x + y - 1) &= 0 \\ 3x + 6y - 15 + 14x + 7y - 7 &= 0 \\ 17x + 13y - 22 &= 0 \end{aligned}$$

and we found, in Chapter 14, that the graph of any equation of the form $Ax + By + C = 0$ is a line, when either A or B is not 0.

Suppose we use this method to find the equation of a line through the intersection of the graphs of the two equations in Problem 1(e) of Problem Set 15-1a:

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

If we have no particular line in mind, we can use any multipliers we wish. Let us choose 3 and -2, and form the equation:

$$3(5x - y + 13) + (-2)(x - 2y - 12) = 0$$

Let us assume that the point (c, d) is the point of intersection of the graphs of the two given equations. Since this point (c, d) is on both graphs, we know that

$$5c - d + 13 = 0 \text{ and } c - 2d - 12 = 0$$

is a true sentence. Why?

Substituting the ordered pair (c, d) in the left side of our new equation, we obtain

$$3(5c - d + 13) + (-2)(c - 2d - 12) = 3(0) + (-2)(0) = 0 .$$

Hence, we know that if the graphs of the first two equations intersect in a point (c, d), the new line also passes through (c, d), even though we do not know what the point (c, d) is.

In general, we can say:

> If $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are the equations of two lines which intersect in exactly one point, and if a and b are real numbers, then
>
> $$a(Ax + By + C) + b(Dx + Ey + F) = 0$$
>
> is the equation of a line which passes through the point of intersection of the first two lines.

Now that we have a method for finding the equations of lines through the intersection of two given lines, let us see if we can select our multipliers a and b with more care, so that we can get the equations of lines parallel to the axes.

The system

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

gave us some trouble when we tried to guess its truth set from the graph. Let us see if this new approach will help us. Form the sentence

$$a(5x - y + 13) + b(x - 2y - 12) = 0.$$

Let us choose a as 2 and b as -1, so that the coefficients of y become opposites:

$$\begin{aligned} (2)(5x - y + 13) + (-1)(x - 2y - 12) &= 0 \\ 10x - 2y + 26 - x + 2y + 12 &= 0 \\ 9x + 38 &= 0 \\ x + 4\tfrac{2}{9} &= 0 \end{aligned}$$

This last equation represents the line through the intersection of the graphs of the two given equations and parallel to the y-axis. Let us go back and select new multipliers that will give us the equation of the line through the intersection point and parallel to the x-axis. What multipliers shall we use? Since we want the coefficients of x to be opposites we choose $a = 1$ and $b = -5$?

$$\begin{aligned}(1)(5x - y + 13) + (-5)(x - 2y - 12) &= 0\\ 5x - y + 13 - 5x + 10y + 60 &= 0\\ 9y + 73 &= 0\\ y + 8\tfrac{1}{9} &= 0\end{aligned}$$

We now have the equations of two new lines, "$x + 4\frac{2}{9} = 0$" and "$y + 8\frac{1}{9} = 0$", each of which passes through the point of intersection of the graphs of the first two equations. Why? This reduces our original problem to finding the point of intersection of these new lines. Can you see what it is? We see, then that the truth set of the system

$$\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$$

is $\left\{\left(-4\frac{2}{9}, -8\frac{1}{9}\right)\right\}$. Verify this fact by showing that these coordinates satisfy both equations.

Now we have a procedure for solving any system of two linear equations. We choose multipliers so as to obtain an equivalent system of lines which are horizontal and vertical. The choice of the multipliers will become easy with practice.

Consider another example: Three times the smaller of two numbers is 6 greater than twice the larger, and three times the larger is 7 greater than four times the smaller. What are the numbers?

The smaller number x and the larger y must satisfy the open sentence

$$3x = 2y + 6 \text{ and } 3y = 4x + 7 .$$

This sentence is equivalent to

$$3x - 2y - 6 = 0 \text{ and } 4x - 3y + 7 = 0 .$$

Choose multipliers so that the coefficients of x will be opposites. 4 and -3 will do the trick.

$$4(3x - 2y - 6) + (-3)(4x - 3y + 7) = 0$$

$$12x - 8y - 24 - 12x + 9y - 21 = 0$$

$$y - 45 = 0$$

Now we could choose multipliers so that the coefficients of y would be opposites. Another way to find the line through the intersection and parallel to the y-axis is as follows: On the line "$y - 45 = 0$" every point has ordinate 45. Thus, the ordinate of the point of intersection is 45. The solution of the sentence "$3x - 2y - 6 = 0$" with ordinate 45 is obtained by solving "$3x - 2(45) - 6 = 0$" or its equivalent, "$x - 32 = 0$". Hence, the sentence "$3x - 2y - 6 = 0$ and $4x - 3y + 7 = 0$" is equivalent to the sentence "$y - 45 = 0$ and $x - 32 = 0$". Now it is easy to read off the solution of the system as $(32, 45)$.

In the above example, what is the solution of the sentence "$4x - 3y + 7 = 0$" with ordinate 45? Does it matter in which sentence we assign the value 45 to y?

Example 1. Find the truth set of

$$\begin{cases} 4x - 3y = 6 \\ 2x + 5y = 16 . \end{cases}$$

$$\begin{cases} 4x - 3y - 6 = 0 \\ 2x + 5y - 16 = 0 \end{cases}$$

$$\begin{aligned} 1(4x - 3y - 6) - 2(2x + 5y - 16) &= 0 \\ 4x - 3y - 6 - 4x - 10y + 32 &= 0 \\ -13y + 26 &= 0 \\ 26 &= 13y \\ 2 &= y \end{aligned}$$

When $y = 2$,

$$\begin{aligned} 4x - 3 \cdot 2 &= 6 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

Therefore "$x = 3$ and $y = 2$" is equivalent to the original sentence.

The truth set is $\{(3, 2)\}$.

Verification:	Left	Right
First clause:	$4 \cdot 3 - 3 \cdot 2 = 12 - 6$	
	$= 6$	6
Second clause:	$2 \cdot 3 + 5 \cdot 2 = 6 + 10$	
	$= 16$	16

Example 2. Solve

$$\begin{cases} 3x = 5y + 2 \\ 2x = 6y + 3 . \end{cases}$$

$$\begin{cases} 3x - 5y - 2 = 0 \\ 2x - 6y - 3 = 0 \end{cases}$$

$$\begin{aligned} 6(3x - 5y - 2) - 5(2x - 6y - 3) &= 0 \\ 18x - 30y - 12 - 10x + 30y + 15 &= 0 \\ 8x + 3 &= 0 \\ 8x &= -3 \\ x &= -\frac{3}{8} \end{aligned}$$

$$\begin{aligned} 2(3x - 5y - 2) - 3(2x - 6y - 3) &= 0 \\ 6x - 10y - 4 - 6x + 18y + 9 &= 0 \\ 8y + 5 &= 0 \\ 8y &= -5 \\ y &= -\frac{5}{8} \end{aligned}$$

Therefore "$x = -\frac{3}{8}$ and $y = -\frac{5}{8}$" is equivalent to the to the original sentence.

The solution is $\left(-\frac{3}{8}, -\frac{5}{8}\right)$.

Verification:	Left	Right
First clause:	$3\left(-\frac{3}{8}\right) = -\frac{9}{8}$	$5\left(-\frac{5}{8}\right) + 2 = -\frac{25}{8} + \frac{16}{8}$ $= -\frac{9}{8}$
Second clause:	$2\left(-\frac{3}{8}\right) = -\frac{3}{4}$	$6\left(-\frac{5}{8}\right) + 3 = -\frac{15}{4} + \frac{12}{4}$ $= -\frac{3}{4}$

Problem Set 15-1b

1. Find the truth sets of the following systems of equations by the method just developed. Draw the graphs of each pair of equations in (a) and (b) with reference to a different set of axes.

(a) $\begin{cases} 3x - 2y - 14 = 0 \\ 2x + 3y + 8 = 0 \end{cases}$ (c) $\begin{cases} 5x - y = 32 \\ x - 2y - 19 = 0 \end{cases}$

(b) $\begin{cases} 5x + 2y = 4 \\ 3x - 2y = 12 \end{cases}$ (d) $\begin{cases} 3x - 2y = 27 \\ 2x - 7y = -50 \end{cases}$

(e) $\begin{cases} x + y - 30 = 0 \\ x - y + 7 = 0 \end{cases}$

(f) $\begin{cases} y = 7x + 5 \\ 4x = y - 3 \end{cases}$

(g) $\begin{cases} 3 - 5x = 0 \\ 3y = x - 6 \end{cases}$

(h) $\begin{cases} \frac{1}{2}x + y = 2 \\ y - \frac{1}{3}x = 1 \end{cases}$

(i) $\begin{cases} 7x - 6y = 9 \\ 9x - 8y = 7 \end{cases}$

2. We can also use the operations on equations stated in Chapter 13 to solve a system of equations. The method which results is essentially the same as that used above. For example, consider the system:

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 8 = 0 \end{cases}$$

and assume that (c, d) is a solution of the system. Then each of the following equations is true.

$$\begin{aligned} 3c - 2d - 5 &= 0 \\ c + 3d - 8 &= 0 \end{aligned}$$

$$\begin{aligned} 3(3c - 2d - 5) &= 3(0) \\ 2(c + 3d - 8) &= 2(0) \end{aligned}$$

$$\begin{aligned} 9c - 6d - 15 &= 0 \\ 2c + 6d - 16 &= 0 \end{aligned}$$

$$11c - 31 = 0$$

$$c = \frac{31}{11}$$

Also,

$$\begin{aligned} 3c - 2d - 5 &= 0 \\ -3(c + 3d - 8) &= -3(0) \end{aligned}$$

$$\begin{aligned} 3c - 2d - 5 &= 0 \\ -3c - 9d + 24 &= 0 \end{aligned}$$

$$-11d + 19 = 0$$

$$d = \frac{19}{11}$$

So _if_ there is a solution of the system

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 8 = 0 \end{cases}$$

then that solution is $(\frac{31}{11}, \frac{19}{11})$. We must verify that this is a solution.

$$3(\tfrac{31}{11}) - 2(\tfrac{19}{11}) - 5 = 0$$

$$\tfrac{31}{11} + 3(\tfrac{19}{11}) - 8 = 0$$

Are these sentences true?

This is often called the _addition method_ of solving systems of equations. Use this method for finding the truth sets of the following systems.

(a) $\begin{cases} x - 4y - 15 = 0 \\ 3x + 5y - 11 = 0 \end{cases}$ (c) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2y \end{cases}$

(b) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2x \end{cases}$ (d) $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 3x \end{cases}$

3. Translate each of the following into open sentences with two variables. Find the truth set of each.

(a) Three hundred eleven tickets were sold for a basketball game, some for pupils and some for adults. Pupil tickets sold for 25 cents each and adult tickets for 75 cents each. The total money received was $108.75. How many pupil and how many adult tickets were sold?

(b) The Boxer family is coming to visit, and no one knows how many children they have. Elsie, one of the girls, says she has as many brothers as sisters; her brother Jimmie says he has twice as many sisters as brothers. How many boys and how many girls are there in the Boxer family?

(c) A home room bought three-cent and four-cent stamps to mail bulletins to the parents. The total cost was \$12.67 . If they bought 352 stamps, how many of each kind were there?

(d) A bank teller has 154 bills of one-dollar and five-dollar denominations. He thinks his total is \$465. Has he counted his money correctly?

4. Find the truth sets of the following compound open sentences. Draw the graphs. Do they help you with (b) and (c) ?

(a) $x - 2y + 6 = 0$ and $2x + 3y + 5 = 0$

(b) $2x - y - 5 = 0$ and $4x - 2y - 10 = 0$

(c) $2x + y - 4 = 0$ and $2x + y - 2 = 0$

5. Find the equation of the line through the intersection of the lines $5x - 7y - 3 = 0$ and $3x - 6y + 5 = 0$ and passing through the origin. (Hint: What is the value of C so that $Ax + By + C = 0$ is a line through the origin?)

In Problem 4 you found some compound open sentences whose truth sets were not single ordered pairs of numbers. Which ones were they? Let us look more closely at each of them.

In the open sentence

$$\text{"}2x - y - 5 = 0 \text{ and } 4x - 2y - 10 = 0\text{"},$$

we note that "$4x - 2y - 10 = 0$" is equivalent to

$$2(2x - y - 5) = 2(0) ;$$

so we see that the graphs of both clauses are identical, as shown in Figure 3, and the lines have many points in common.

State some of the numbers of the truth set of the compound sentence. Is the truth set a finite set? What happened when you tried to solve the open sentence algebraically? Why didn't the method work?

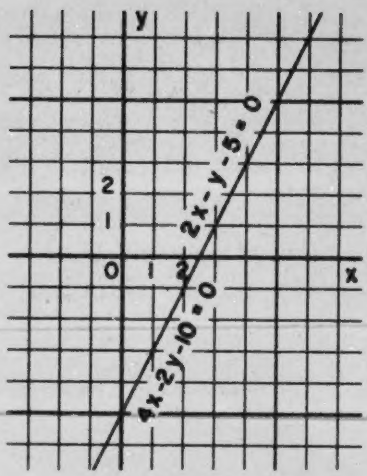


Figure 3.

A somewhat different condition exists in the compound sentence "$2x + y - 4 = 0$ and $2x + y - 2 = 0$". Putting each clause into the y-form, we have

$$y = -2x + 4 \quad \text{and} \quad y = -2x + 2 \ .$$

What is the slope of the graph of each of these equations? What is the y-intercept number? We see that the graphs are two parallel lines, as in Figure 4, and there is no intersection point. In such a case, the truth set of the compound sentence is the null set. What happens if we try to solve the sentence algebraically? Why?

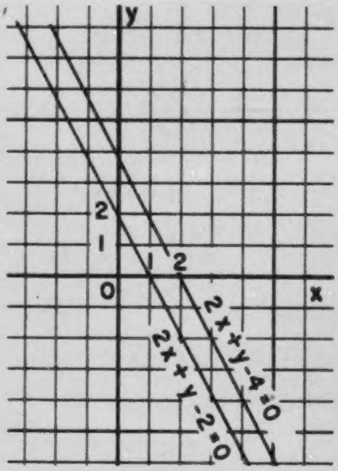


Figure 4.

[sec. 15-1]

Let us try to summarize what we have noted here: The truth set of a compound open sentence in two variables, with connective "and", may consist of one ordered pair, many ordered pairs, or no ordered pairs. Correspondingly, the graphs of the two clauses of the open sentence may have one intersection, many intersections, or no intersections.

Example 1.

Equations	Graphs
$2x - 3y = 4$ and $x + y = 7$ $y = \frac{2}{3}x - \frac{4}{3}$ and $y = -x + 7$ The truth set is $\{(5, 2)\}$.	The two lines which are the graphs of the clauses have one intersection, since the slopes of the lines are not the same. The graph of the truth set is the single point $(5, 2)$.
Example 2.	
$2x - 3y = 7$ and $4x - 6y = 14$ $y = \frac{2}{3}x - \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{14}{6}$ The truth set is made up of all the ordered pairs whose coordinates satisfy the first equation. (Note that the second clause is obtained if each member of the first clause of the original open sentence is multiplied by 2.)	The graphs of the two clauses of the open sentence coincide, since the lines have the same slope and the same y-intercept number. The entire line is the graph of the truth set.
Example 3.	
$2x - 3y = 7$ and $4x - 6y = 3$ $y = \frac{2}{3}x - \frac{7}{3}$ and $y = \frac{4}{6}x - \frac{3}{6}$ The truth set is $\emptyset$.	The graphs of the two clauses of the open sentence are parallel lines, because they have the same slope but different y-intercept numbers. The graph of the truth set contains no points.

Notice, in Example 3, that the coefficients of x and y in the equation $2x - 3y = 7$ are related in a simple way to those in the equation $4x - 6y = 3$.

$$2 = \frac{1}{2}(4) \text{ and } -3 = \frac{1}{2}(-6).$$

In general, the real numbers A and B are said to be proportional to the real numbers C and D if there is a real number k, other than 0, such that

$$A = kC \text{ and } B = kD.$$

Thus, the numbers 2 and -3 are proportional to 4 and -6, the number k being $\frac{1}{2}$. If two lines are parallel, what can you say about the coefficients of x and y in their equations?

Another way to say that A and B are proportional to C and D is to say that the ratios

$$\frac{A}{C} \text{ and } \frac{B}{D}$$

are equal, or

$$\frac{A}{C} = \frac{B}{D}.$$

Problem Set 15-1c

1. Draw the graphs of the open sentences in Examples 1 to 3 on page 479. Find the truth set of Example 1 algebraically.
2. Solve the following compound open sentences. Draw the graphs in (a) and (b).
   - (a) $3x + 4y - 13 = 0$ and $5x - 2y + 13 = 0$
   - (b) $x + 5y - 17 = 0$ and $2x - 3y - 8 = 0$
   - (c) $5x - 4y + 2 = 0$ and $10x - 8y + 4 = 0$
   - (d) $12x - 4y - 5 = 0$ and $6x + 8y - 5 = 0$
   - (e) $x - 2y - 5 = 0$ and $3x - 6y - 12 = 0$
   - (f) $\frac{1}{3}(\frac{6x}{7} - \frac{3y}{5}) - 1 = 0$ and $\frac{2}{3}(\frac{4x}{7} + \frac{y}{10}) - \frac{7}{3} = 0$

3. Consider the system,

$$\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$$

Suppose we write the y-form for each equation and draw its graph

$$y = 2x - 7 \quad \text{and} \quad y = -\frac{5}{2}x + 2$$

At what point on the graph of this system are the values of y equal? What is the value of x at this point? If we set the values of y in the two sentences equal to each other, we have the open sentence in one variable,

$$2x - 7 = -\frac{5}{2}x + 2 .$$

The truth set of this sentence is $\{2\}$. Using this value for x in each open sentence which is in the y-form we get:

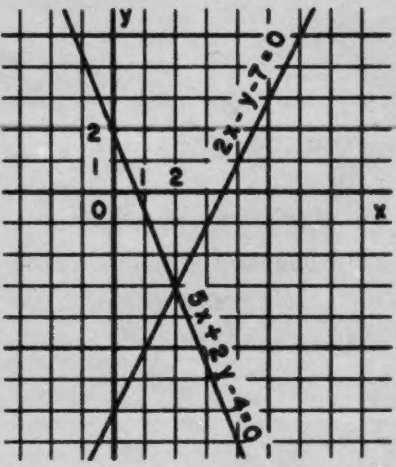


Figure 5.

$$y = 2(2) - 7 \qquad\qquad y = -\frac{5}{2}(2) + 2$$

$$y = -3 \qquad\qquad y = -3$$

Why do we get "$y = -3$" in both cases? Hence, if the compound open sentence "$2x - y - 7 = 0$ and $5x + 2y - 4 = 0$" has a solution, it must be $(2, -3)$. Verify that $(2, -3)$ is the solution.

Suppose that we shorten our work somewhat by writing only the first equation in its y-form.

$$y = 2x - 7$$

Then we replace y in the second equation by the expression "$2x - 7$".

$$5x + 2(2x - 7) - 4 = 0$$

Let us proceed to solve this open sentence in one variable

$$5x + 4x - 14 - 4 = 0$$
$$9x - 18 = 0$$
$$x = 2 .$$

Then,

$$y = 2x - 7 = 2(2) - 7 = -3$$

so that $(2, -3)$ is the possible solution of the system

$$\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 . \end{cases}$$

The method just described in which we "solve one of the equations for y in terms of x" and then substitute this expression for y into the other equation is called a substitution method. Solve the following systems using which ever of the above methods seems more appropriate.

(a) $\begin{cases} 3x + y + 18 = 0 \\ 2x - 7y - 34 = 0 \end{cases}$

(b) $\begin{cases} y = \frac{2}{3}x + 2 \\ y = -\frac{5}{2}x + 40 \end{cases}$

(c) $\begin{cases} 5x + 2y - 4 = 0 \\ 10x + 4y - 8 = 0 \end{cases}$

(d) $\begin{cases} 5x + 2y - 5 = 0 \\ x - 3y - 18 = 0 \end{cases}$

(e) $\begin{cases} x + 7y = 11 \\ x - 3y = -4 \end{cases}$

(f) $\begin{cases} x = \frac{3}{2}y - 4 \\ y = -\frac{2}{3}x \end{cases}$

4. As we have seen, the truth set of the compound open sentence

$$Ax + By + C = 0 \quad \text{and} \quad Dx + Ey + F = 0$$

may consist of one ordered pair of numbers, of many ordered pairs, or of no ordered pairs.

(a) If the truth set consists of one ordered pair, what can you say about the graphs of the clauses?

(b) If the truth set consists of many ordered pairs, what is true of the graphs of the two clauses? Are the two clauses of the compound sentence equivalent?

(c) If the truth set is $\emptyset$, how are the coefficients of x and y related in the two clauses? What is true of the graphs of the clauses?

5. Consider the system

$$\begin{cases} 4x + 2y - 11 = 0 \\ 3x - y - 2 = 0 \end{cases}$$

In four different ways find its truth set.

6. Solve in any way. Explain why you chose a particular method in each case.

(a) $\begin{cases} 3x + 2y = 1 \\ 2x - 3y = 18 \end{cases}$

(b) $\begin{cases} x - 2y = 0 \\ x + 2y = 0 \end{cases}$

(c) $\begin{cases} x = 2y - \frac{1}{6} \\ 2x + y = \frac{4}{3} \end{cases}$

(d) $\begin{cases} 3x - 4y - 1 = 0 \\ 7x + 4y - 9 = 0 \end{cases}$

(e) $\begin{cases} \frac{x}{2} - \frac{x}{3} = 1 \\ x + y = 7 \end{cases}$

(f) $\begin{cases} 6y + (2 - 4x) = 3 \\ 4x - 2(3y - 1) = 2 \end{cases}$

(g) $\begin{cases} 5 - (x + y) = 2y \\ 2x - (3y + 1) = 1 \end{cases}$

(h) $\begin{cases} 7x - y = 28 \\ 3x + 11y = 92 \end{cases}$

In Problems 7 - 17 translate into open sentences, find the truth set and answer the question asked.

7. Find two numbers whose sum is 56 and whose difference is 18.

8. The sum of Sally's and Joe's ages is 30 years. In five years the difference of their ages will be 4 years. What are their ages now?

9. A dealer has some cashew nuts that sell at \$1.20 a pound and almonds that sell at \$1.50 a pound. How many pounds of each should he put into a mixture of 200 pounds to sell at \$1.32 a pound?

10. In a certain two digit number the units' digit is one more than twice the tens' digit. If the units' digit is added to the number, the sum is 35 more than three times the tens' digit. Find the number.

11. Hugh weighs 80 pounds and Fred weighs 100 pounds. They balance on a teeterboard that is 9 feet long. Each sits at an end of the board. How far is each boy from the point of balance?

12. Two boys sit on a see-saw, one five feet from the fulcrum (the point where it balances), the other on the other side six feet from the fulcrum. If the sum of the boys' weights is 209 pounds, how much does each boy weigh?

13. It takes a boat $1\frac{1}{2}$ hours to go 12 miles down stream, and 6 hours to return. Find the speed of the current and the speed of the boat in still water.

14. Three pounds of apples and four pounds of bananas cost $1.08, while 4 pounds of apples and 3 pounds of bananas cost $1.02. What is the price per pound of apples? Of bananas?

15. A and B are 30 miles apart. If they leave at the same time and walk in the same direction, A overtakes B in 60 hours. If they walk toward each other they meet in 5 hours. What are their speeds?

16. A 90% solution of alcohol is to be mixed with a 75% solution to make 20 quarts of a 78% solution. How many quarts of the 90% solution should be used?

17. In a 300 mile race the driver of car A gives the driver of car B a start of 25 miles, and still finishes one-half hour sooner. In a second trial, the driver of car A gives the driver of car B a start of 60 miles and loses by 12 minutes. What were the average speeds of cars A and B in miles per hour?

15-2. Systems of Inequalities

In 15-1 we defined a system of equations as a compound open sentence in which two equations are joined by the connective "and". We also introduced a notation for this. Carrying the idea over to inequalities, let us consider systems like the following:

(a) $\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$ (b) $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$

(c) What would the graph of $x + 2y - 4 > 0$ be? You recall that we first draw the graph of

$$x + 2y - 4 = 0,$$

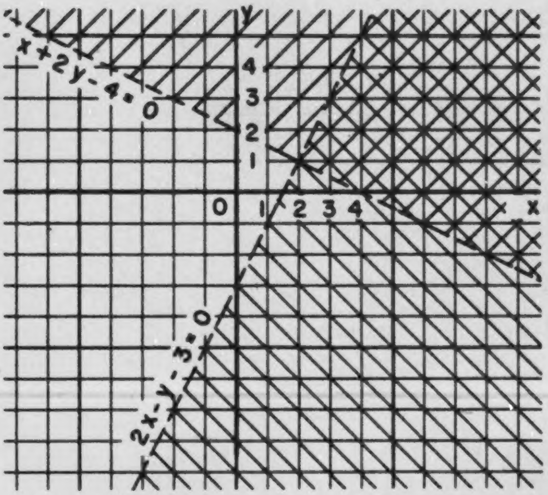


Figure 6.

using a dashed line along the boundary. Why? Then we shade the region above the line, since the graph of "$x + 2y - 4 > 0$", i.e. of "$y > -\frac{1}{2}x + 2$", consists of all those points whose ordinate is greater than "two more than $-\frac{1}{2}$ times the abscissa". In a similar way, we shade the region where "$y < 2x - 3$". This is the region below the line whose equation is "$2x - y - 3 = 0$". Why is the line here also dashed? When would we use a solid line as boundary?

Since the truth set of a compound open sentence with the connective and is the set of elements common to the truth sets of the two clauses, it follows that the truth set of the system (a) is the region indicated by double shading in Figure 6.

(b) What would be the graph of a system in which we have one equation and one inequality, such as Example (b)? What is the graph of "$3x - 2y - 5 = 0$"?

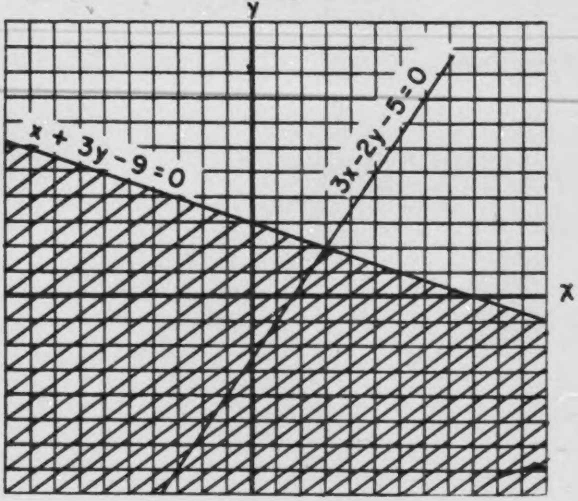


Figure 7.

Is the graph of "$x + 3y - 9 \leq 0$" the region above or below the line

$$x + 3y - 9 = 0 \text{ ?}$$

Is the line included? Study Figure 7 carefully, and describe the graph of the system

$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$$

Problem Set 15-2a

Draw graphs of the truth sets of the following systems:

1. $\begin{cases} 2x + y > 8 \\ 4x - 2y \leq 4 \end{cases}$

2. $\begin{cases} 6x + 3y < 0 \\ 4x - y < 6 \end{cases}$

3. $\begin{cases} 5x + 2y + 1 > 0 \\ 3x - y - 6 = 0 \end{cases}$

4. $\begin{cases} 4x + 2y = -1 \\ y - x \geq 4 \end{cases}$

5. $\begin{cases} 2x + y < 4 \\ 2x + y > 6 \end{cases}$

6. $\begin{cases} 2x + y > 4 \\ 2x + y < 6 \end{cases}$

7. $\begin{cases} 2x - y \leq 4 \\ 4x - 2y < 8 \end{cases}$

Let us consider the graph of the compound open sentence

$$x - y - 2 > 0 \text{ or } x + y - 2 > 0 .$$

First we draw the graphs of the clauses "$x - y - 2 > 0$" and "$x + y - 2 > 0$".

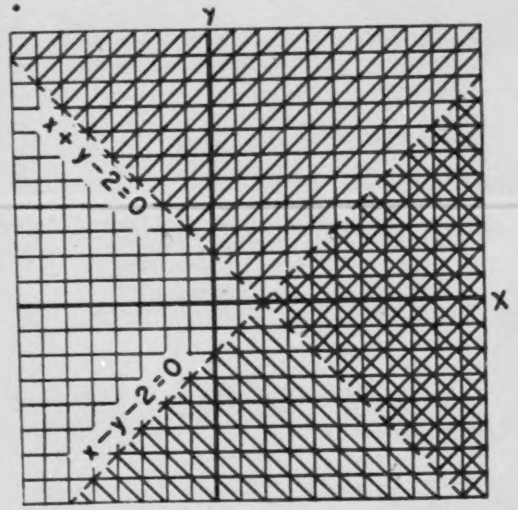


Figure 8.

Next we recall that the truth set of a compound open sentence with the connective or is the set of all elements in either of the truth sets of the clauses. Hence, the graph of the compound open sentence under consideration includes the entire shaded region in Figure 8.

Problem Set 15-2b

Draw the graphs of the truth sets of the following sentences:

1. $2x + y + 3 > 0$ or $3x + y + 1 < 0$
2. $2x + y + 3 < 0$ or $3x - y + 1 < 0$
3. $2x + y + 3 \leq 0$ or $3x + y + 1 \geq 0$
4. $2x + y + 3 > 0$ and $3x - y + 1 < 0$

To complete the picture, let us consider the compound open sentence:

$$(x - y - 2)(x + y - 2) > 0 .$$

Remember that "$ab > 0$" means that "the product of a and b is a positive number". What can be said of a and b if $ab > 0$? Thus we have the two possibilities:

$$x - y - 2 > 0 \text{ and } x + y - 2 > 0 ,$$

or

$$x - y - 2 < 0 \text{ and } x + y - 2 < 0 .$$

In Figure 8, the graph of "$x - y - 2 > 0$ and $x + y - 2 > 0$" is the region indicated by double shading, while the graph of "$x - y - 2 < 0$ and $x + y - 2 < 0$" is the unshaded region. So the graph of

$$(x - y - 2)(x + y - 2) > 0$$

consists of all the points in these two regions of the plane.

Which areas form the graph of the open sentence

$$(x - y - 2)(x + y - 2) < 0 ?$$

(If $ab < 0$, what can be said of a and b ?)

To summarize, we list the following pairs of equivalent sentences (a and b are real numbers):

$ab = 0$: $a = 0$ or $b = 0$.

$ab > 0$: $a > 0$ and $b > 0$, or $a < 0$ and $b < 0$.

$ab < 0$: $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$.

Verify these equivalences by going back to the definition of the product of real numbers.

Problem Set 15-2c

1. Draw the graphs of the truth sets of the following open sentences.

   (a) $(2x - y - 2)(3x + y - 3) > 0$

   (b) $(x + 2y - 4)(2x - y - 3) < 0$

   (c) $(x + 2y - 6)(2x + 4y + 4) > 0$

   (d) $(x - y - 3)(3x - 3y - 9) < 0$

2. Draw the graphs of the truth sets of the following open sentences.

   (a) $x - 3y - 6 = 0$ and $3x + y + 2 = 0$

   (b) $(x - 3y - 6)(3x + y + 2) = 0$

   (c) $x - 3y - 6 > 0$ and $3x + y + 2 > 0$

   (d) $x - 3y - 6 < 0$ and $3x + y + 2 < 0$

   (e) $x - 3y - 6 > 0$ and $3x + y + 2 = 0$

   (f) $x - 3y - 6 < 0$ or $3x + y + 2 < 0$

   (g) $x - 3y - 6 = 0$ or $3x + y + 2 \geq 0$

   (h) $(x - 3y - 6)(3x + y + 2) > 0$

   (i) $(x - 3y - 6)(3x + y + 2) < 0$

3. Draw the graph of the truth set of each of these systems of inequalities. (The brace again indicates a compound sentence with connective and.)

   (a) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 3x + 4y \leq 12 \end{cases}$

   (c) $\begin{cases} -4 < x < 4 \\ -3 < y < 3 \end{cases}$

   (b) $\begin{cases} y \geq 2 \\ 4y \leq 3x + 8 \\ 4y + 5x \leq 40 \end{cases}$

*4. A football team finds itself on its own 40 yard line, in possession of the ball, with five minutes left in the game. The score is 3 to 0 in favor of the opposing team. The quarterback knows the team should make 3 yards on each running play, but will use 30 seconds per play. He can make 20 yards on a successful pass play, which uses 15 seconds. However, he usually completes only one pass out of three. What combination of plays will assure a victory, or what should be the strategy of the quarterback?

Review Problems

1. Find the truth set of "$2x - 3 = 0$" and draw its graph if it is considered as an equation in
   (a) one variable, (b) two variables.

2. Find the truth set of "$|y| < 3$" and draw its graph if it is considered as a sentence in
   (a) one variable, (b) two variables.

3. For each of the following pairs of equations decide whether the two equations are equivalent, and tell why.
   (a) $x^2 = 3x - 2$; $x - 2 = 0$ and $x - 1 = 0$
   (b) $\frac{2x}{x^2 + 1} = 1$; $x^2 - 2x + 1 = 0$
   (c) $xy > 0$; $x > 0$ or $y > 0$
   (d) $\frac{y - 1}{x - 1} = 2$; $y - 1 = 2(x - 1)$
   (e) $x + y = 3$ and $\frac{x}{2} - \frac{y}{3} = 4$; $x - 6 = 0$ and $y + 3 = 0$.

4. Given the lines with equations $3x - 5y - 4 = 0$ and $2x + 3y + 4 = 0$. What are the equations of two lines each of which contains the point of intersection of the two given lines, one of which is vertical and the other horizontal?

5. Solve the following systems and in each case tell why you chose a particular method.

   (a) $\begin{cases} 2x = 3y + 1 \\ 4x - 3y = 11 \end{cases}$ (d) $\begin{cases} x = 9y \\ \frac{1}{3}x = 3y + 2 \end{cases}$

   (b) $\begin{cases} .01x - .02y = 0 \\ x - 10y = 8 \end{cases}$ (e) $\begin{cases} .5x = .2y + 4 \\ y - x = -11 \end{cases}$

   (c) $\begin{cases} y = 2x - 4 \\ x - \frac{1}{2}y - 2 = 0 \end{cases}$ (f) $\begin{cases} \frac{1}{8}x - \frac{1}{3}y = 0 \\ \frac{1}{6}x - \frac{1}{9}y = 3 \end{cases}$

6. (a) Discuss the relationship among the coefficients of the equations of two parallel lines.

   (b) Discuss the positions of two lines if their equations are $Ax + By + C = 0$ and $Dx + Ey + F = 0$ and

   $$\frac{A}{D} = \frac{B}{E} = \frac{C}{F} .$$

   (c) Describe the conditions on the slopes of two lines which guarantee that the lines have exactly one common point.

7. Draw the graphs of the following sentences.

   (a) $y + 3x - 2 > 0$

   (b) $2x - 3y + 3 > 0$

   (c) $y + 3x - 2 > 0$ and $2x - 3y + 3 > 0$

   (d) $(y + 3x - 2)(2x - 3y + 3) < 0$

   (e) $|y + 3x| > 2$

8. Translate the following into open sentences and solve.

   (a) Find two consecutive integers whose sum is 57.

   (b) Find two integers such that their sum is 16 and twice the first is three less than the second.

(c) The sum of two numbers is 45. If the larger is divided by the smaller, the quotient is 4 and the remainder is 5. What are the numbers?

(d) Two grades of tobacco are mixed, the one selling for $4.80 per pound and the other for $6.00 per pound. How many pounds of each grade must be blended to obtain 20 pounds of a mixture to sell for $5.50 per pound?

Chapter 16

QUADRATIC POLYNOMIALS

16-1. *Graphs of Quadratic Polynomials*

In Chapter 12 we first studied *quadratic polynomials*, that is, polynomials in one variable which involve the square but no higher powers of the variable. Every such polynomial can be written in the form

$$Ax^2 + Bx + C,$$

where A, B, and C are real numbers with $A \neq 0$. Is "$-2(x + 1)^2 + 3$" a quadratic polynomial? By the *graph of the polynomial* $Ax^2 + Bx + C$ we mean the graph of the open sentence

$$y = Ax^2 + Bx + C.$$

We can make a drawing of the graph of a quadratic polynomial by locating some of the points of the graph.

Example 1. Draw the graph of the polynomial

$$x^2 - 2x - 3.$$

Let us list some ordered pairs satisfying the equation

$$y = x^2 - 2x - 3.$$

x	-2	$-\frac{3}{2}$	-1	$-\frac{2}{3}$	0	$\frac{1}{2}$	1	$\frac{4}{3}$		$\frac{5}{2}$	3	4
y	5	$\frac{9}{4}$			-3		-4		-3	$-\frac{7}{4}$		

Fill in the missing numbers in this table and then locate these points with reference to a set of coordinate axes. The arrangement of the points suggests that the graph might look like the one sketched in Figure 1.

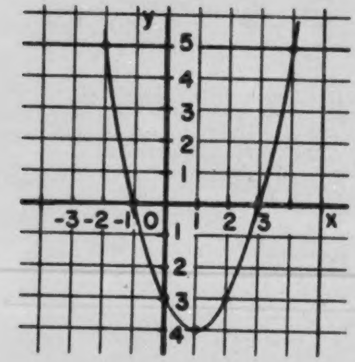


Figure 1.

By locating more points whose coordinates satisfy the equation you can convince yourself that the graph does indeed have the indicated shape. A more systematic discussion of the shape of such graphs will be found in Chapter 17.

Problem Set 16-1a

Draw the graphs of the polynomials.

1. $2x^2$, for x between -2 and 2 .
2. $x^2 - 2$, for x between -3 and 3 .
3. $-\frac{1}{2}x^2 + x$, for x between -3 and 3 .
4. $x^2 + x + 1$, for x between -3 and 2 .
5. $x^2 - 4x + 4$, for x between -1 and 5 .
6. $2x^2 - 3x - 5$, for x between -2 and 3 .

You probably noticed that the preceding problems took a good deal of time and effort. Even then, you were only guessing at the shapes of the graphs. Let us try to develop a more precise method for drawing such graphs.

Start with the most simple quadratic polynomial, "x^2". (In previous sections we located some points on the graph of "$y = x^2$".) Then let us see how this graph differs from that of "$y = \frac{1}{2}x^2$", of "$y = 2x^2$", of "$y = -\frac{1}{2}x^2$". In general, what will be the shape of the graph of

$$y = ax^2,$$

where a is a non-zero real number? If we draw all these graphs with reference to one set of axes, we will be able to compare them. A list of values of these polynomials for given values of x is as follows:

x	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	-0.1	0	$\frac{1}{3}$	1	$\frac{4}{3}$	2	
x^2		4		1	$\frac{1}{4}$		0		1		4	
$2x^2$		8		2	$\frac{1}{2}$		0		2		8	18
$\frac{1}{2}x^2$		2		$\frac{1}{2}$	$\frac{1}{8}$		0		$\frac{1}{2}$		2	
$-\frac{1}{2}x^2$		-2		$-\frac{1}{2}$	$-\frac{1}{8}$		0		$-\frac{1}{2}$		-2	

You fill in the missing numbers. The graphs in Figure 2 are suggested by this table.

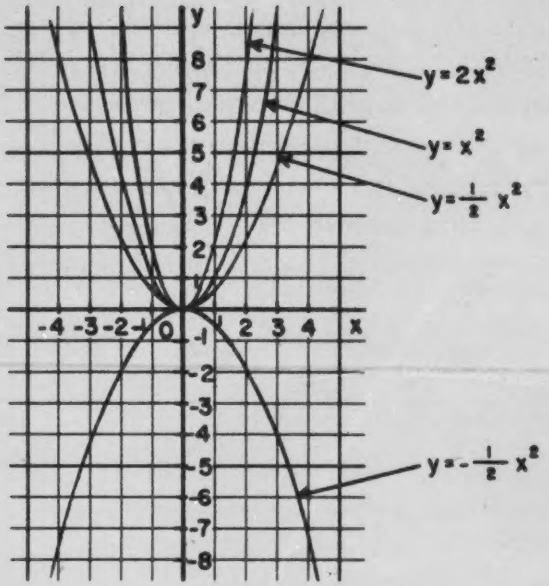


Figure 2.

Problem Set 16-1b

1. How can you obtain the graph of "$2x^2$" from the graph of "x^2" ?

2. How can you obtain the graph of " $-\frac{1}{2}x^2$ " from the graph of "$\frac{1}{2}x^2$" ?

3. Draw the graph of "$5x^2$" for x between -1 and 1 .

4. Draw the graph of "$\frac{1}{5}x^2$" for x between -10 and 10 .

5. How can you obtain the graph of "$-5x^2$" from the graph of "$5x^2$" ?

6. Explain how you can obtain the graph of "$-ax^2$" from the graph of "ax^2", where a is any non-zero real number.

Now that we have a graph of the polynomial "ax^2", for any non-zero number a, let us move this graph horizontally to obtain graphs of other quadratic polynomials. As an example, let us draw the graph of

$$\frac{1}{2}(x - 3)^2$$

and see how it can be obtained from the graph of "$\frac{1}{2}x^2$". Let us list a table of coordinates satisfying the equation

$$y = \frac{1}{2}(x - 3)^2 .$$

x	0	$\frac{1}{3}$	1	$\frac{3}{2}$	2	2.5		$\frac{13}{4}$	4	5
y		$\frac{32}{9}$		$\frac{9}{8}$			0	$\frac{1}{32}$		

Fill in the missing numbers. The graph is compared in Figure 3 with the graph of "$y = \frac{1}{2}x^2$" .

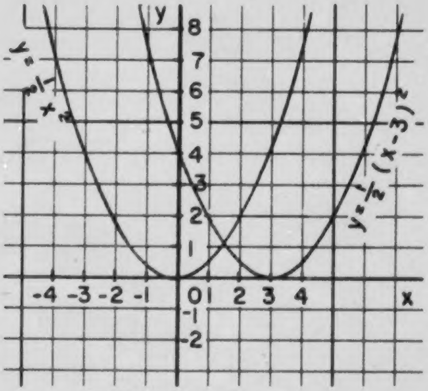


Figure 3.

We notice that the graph of $y = \frac{1}{2}(x - 3)^2$ has exactly the same shape as the graph of

$$y = \frac{1}{2}x^2 ,$$

but is 3 units to the right. In the same way we could verify that the graph of "$y = 2(x + 2)^2$" is 2 units to the left of the graph of "$y = 2x^2$" and has the same shape as "$y = 2x^2$". How could we obtain the graph of

$$y = -(x + 3)^2$$

from the graph of

$$y = -x^2 \quad ?$$

Problem Set 16-1c

1. After setting up a table of coordinates of points, draw carefully the graph of

   $$y = 2(x + 2)^2 \;;$$

   with reference to the same coordinate axes draw the graph of

   $$y = 2x^2 \;.$$

   From the figure describe how you can obtain the graph of "$y = 2(x + 2)^2$" from the graph of "$y = 2x^2$".

2. For each of the following, describe how you can obtain the graph of the first from the graph of the second equation.

   (a) $y = 3(x + 4)^2 \quad ; \quad y = 3x^2$

   (b) $y = -2(x - 3)^2 \quad ; \quad y = -2x^2$

   (c) $y = -\frac{1}{2}(x + 1)^2 \quad ; \quad y = -\frac{1}{2}x^2$

   (d) $y = \frac{1}{3}(x + \frac{1}{2})^2 \quad ; \quad y = \frac{1}{3}x^2$

3. Give a general rule for obtaining the graph of "$y = a(x - h)^2$" from the graph of "$y = ax^2$", where a and h are real numbers and $a \neq 0$.

Next, let us move the graphs of polynomials vertically. Consider the quadratic polynomial,

$$\frac{1}{2}(x - 3)^2 + 2 \text{ ,}$$

and compare it with the graph which we have already obtained, of "$\frac{1}{2}(x - 3)^2$" . A table of coordinates satisfying "$y = \frac{1}{2}(x - 3)^2+2$" is the following:

x	0	$\frac{1}{3}$	1	$\frac{3}{2}$	2	2.5	3	$\frac{13}{4}$	4	5
y		$\frac{50}{9}$		$\frac{25}{8}$			2	$\frac{65}{32}$		4

(You have probably observed that each ordinate in this table is 2 greater than the corresponding ordinate in the preceding table.)

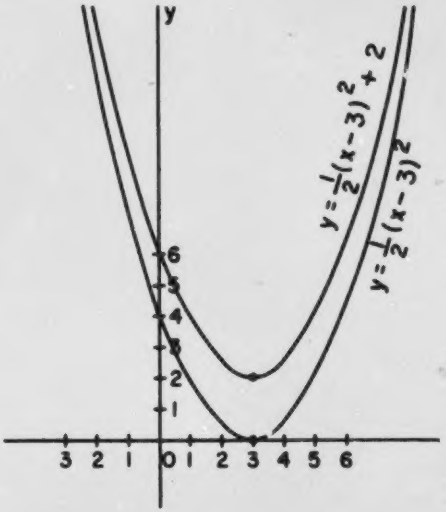


Figure 4.

Again we observe that the shape of the graph has not changed, but that the graph of

$$y = \frac{1}{2}(x - 3)^2 + 2$$

is obtained by moving the graph of

$$y = \frac{1}{2}(x - 3)^2$$

upward 2 units. Similarly, we can show that the graph of "$y = 2(x + 2)^2 - 3$" can be obtained by moving the graph of "$y = 2(x + 2)^2$" downward 3 units.

Finally, we notice that the graphs in Figures 3 and 4 are exactly the same shape, and that we can obtain the graph "$y = \frac{1}{2}(x - 3)^2 + 2$" by moving the graph of "$y = \frac{1}{2}x^2$" to the right 3 units and upward 2 units.

We shall see in Chapter 17 that it is always possible to obtain the graph of

$$y = a(x - h)^2 + k$$

from the graph of

$$y = ax^2$$

by moving the graph of "$y = ax^2$" horizontally h units and vertically k units.

These graphs (of quadratic polynomials) are called parabolas. The lowest (or highest) point on the graph is called the vertex, and the vertical line through the vertex is called the axis. Thus, the vertex of the parabola whose equation is

$$y = 2x^2$$

is $(0, 0)$ and its axis is the line with the equation $x = 0$. What are the vertex and axis of the parabola whose equation is

$$y = \frac{1}{2}(x - 3)^2 + 2 \text{ ?}$$

[set 16-1]

Problem Set 16-1d

1. Describe how the graphs of "$y = x^2 - 3$" and "$y = x^2 + 3$" can be obtained from the graph of "$y = x^2$" ? Draw all three graphs with reference to the same axes.

2. How can the graph of "$y = 2(x - 2)^2 + 3$" be obtained from the graph of "$y = 2x^2$" ? Draw both graphs with reference to the same axes.

3. Draw the parabola whose equation is "$y = (x + 1)^2 - \frac{1}{2}$" . Describe how you obtain this graph from the graph of "$y = x^2$" . What are the coordinates of its vertex and the equation of its axis?

4. Draw the parabola whose equation is "$y = -2(x + \frac{1}{2})^2 + 3$" . How can this parabola be obtained from the graph of "$y = -2x^2$" ?

5. Find equations for the following parabolas.

   (a) The graph of "$y = x^2$" moved 5 units to the left and 2 units downward.

   (b) The graph of "$y = -x^2$" moved 2 units to the left and 3 units upward.

   (c) The graph of "$y = \frac{1}{3}x^2$" moved $\frac{1}{2}$ unit to the right and 1 unit downward.

   (d) The graph of "$y = \frac{1}{2}(x + 7)^2 - 4$" moved 7 units to the right and 4 units upward.

6. Describe, without drawing, the graph of each of the following.

   (a) $y = 3(x - 2)^2 - 4$

   (b) $y = - (x + 3)^2 + 1$

   (c) $y = \frac{1}{2}(x - 2)^2 - 2$

   (d) $y = -2(x + 1)^2 + 2$

16-2. Standard Forms

We have learned how to obtain the graph of "$y = (x - 1)^2 - 4$" quickly. This is the parabola obtained by moving the graph of "$y = x^2$" 1 unit to the right and 4 units downward. We also notice that $(x - 1)^2 - 4 = x^2 - 2x - 3$, for every real number x. Therefore, we have obtained the graph of the equation

$$y = x^2 - 2x - 3 .$$

Suppose we were given the equation in the form "$y = x^2 - 2x - 3$" instead of "$y = (x - 1)^2 - 4$". How would we go about finding this second form? The fact to notice is that in the second form, x is involved only in an expression which is a perfect square. Therefore, we might ask ourselves the question: How can $x^2 - 2x - 3$ be changed into a form in which x is involved only in a perfect square? Do you remember this problem in Chapter 12 in connection with factoring?

We can do this by "working backward" from "$x^2 - 2x - 3$" as follows:

$$x^2 - 2x - 3 = (x^2 - 2x \quad) - 3 .$$

Now we ask: What is missing inside the parentheses to yield a perfect square? Clearly, what is needed is a "1". Why? Then we have

$$\begin{aligned} x^2 - 2x - 3 &= (x^2 - 2x + 1) - 3 - 1 , \\ &= (x - 1)^2 - 4 . \end{aligned}$$

Why did we also add "-1" as well as "1" ?

Let us follow the same procedure with the polynomial "$3x^2 - 12x + 5$". We have

$$\begin{aligned} 3x^2 - 12x + 5 &= 3(x^2 - 4x \quad) + 5 , \\ &= 3(x^2 - 4x + 4) + 5 - (3)(4) , \\ &= 3(x - 2)^2 - 7 . \end{aligned}$$

Hence, the graph of

$$y = 3x^2 - 12x + 5$$

is a parabola with vertex $(2, -7)$ and axis $x = 2$. It is obtained by moving the graph of "$y = 3x^2$" to the right 2 units and downward 7 units.

As you may have already recalled from Chapter 12, this method of writing a quadratic polynomial in a form in which the variable is involved only in a perfect square is called *completing the square*. The resulting form is called the *standard form* of the quadratic polynomial.

Problem Set 16-2

1. Put each of the following quadratic polynomials in standard form.

   (a) $x^2 - 2x$
   (b) $x^2 + x + 1$
   (c) $x^2 + 6x$
   (d) $x^2 - 3x - 2$
   (e) $x^2 - 3x + 2$
   (f) $5x^2 - 10x - 5$
   (g) $4x^2 + 4$
   (h) $x^2 + kx$, k a real number
   (i) $x^2 + \sqrt{2}x - 1$
   (j) $\frac{1}{2}x^2 - 3x + 2$

2. Put each of the following quadratic polynomials in standard form.

   (a) $x^2 - x + 2$
   (b) $x^2 + 3x + 1$
   (c) $3x^2 - 2x$
   (d) $(x + 5)(x - 5)$
   (e) $6x^2 - x - 15$
   (f) $(x + 1 - \sqrt{2})(x + 1 + \sqrt{2})$

3. Describe, without drawing the graphs, the parabolas which are the graphs of the polynomials in Problem 2.

4. Draw the graph of

$$y = x^2 + 6x + 5 .$$

In how many points does it cross the x-axis? What points?

5. Draw the graph of

$$y = x^2 + 6x + 9 .$$

In how many points does it cross the x-axis? What points?

6. Draw the graph of

$$y = x^2 + 6x + 13 .$$

In how many points does it cross the x-axis?

7. Solve the equations formed in Problems 4 and 5 by letting y have the value 0. Compare the truth sets of these equations with the points where the parabolas cross the x-axis. Devise a rule for determining the points in which a parabola crosses the x-axis.

8. Consider the standard forms of the quadratic polynomials in Problems 4, 5 and 6. Which of these polynomials can be factored as the difference of two squares?

16-3. Quadratic Equations

In Problem 7 we learned that the graph of the parabola

$$y = Ax^2 + Bx + C$$

crosses the x-axis at points whose abscissas satisfy the equation

$$Ax^2 + Bx + C = 0 .$$

This is called a quadratic equation if $A \neq 0$.

We solved such quadratic equations before in the cases where "$Ax^2 + Bx + C$" could be factored as a polynomial over the integers. Can "$2x^2 - 3x + 1$" be factored as a polynomial over the integers? If so, review how you solved the equation

$$2x^2 - 3x + 1 = 0 .$$

Now we can go a step farther, because we can write any quadratic polynomial in standard form.

Example 1. Solve the equation

$$x^2 - 2x - 2 = 0 .$$

First, we write the polynomial in standard form:

$$x^2 - 2x - 2 = (x - 1)^2 - 3 .$$

Since $3 = (\sqrt{3})^2$, we may treat the polynomial as the difference of two squares:

$$x^2 - 2x - 2 = (x - 1)^2 - (\sqrt{3})^2 .$$

Next, recall how to factor the difference of two squares:

$$(x - 1)^2 - (\sqrt{3})^2 = \Big((x - 1) + \sqrt{3}\Big)\Big((x - 1) - \sqrt{3}\Big).$$

Now we have factored the polynomial over the real numbers:

$$x^2 - 2x - 2 = \left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)$$

Multiply these factors and verify the product. The final step in the solution is the familiar process of writing the equation "ab = 0" in its equivalent form "a = 0 or b = 0", where a and b are real numbers. Then the sentences

$$x^2 - 2x - 2 = 0$$

$$\left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right) = 0$$

$$x - 1 + \sqrt{3} = 0 \quad \text{or} \quad x - 1 - \sqrt{3} = 0$$

$$x = 1 - \sqrt{3} \quad \text{or} \quad x = 1 + \sqrt{3}$$

are all equivalent. Hence, the truth set of the equation is $\{1 - \sqrt{3},\ 1 + \sqrt{3}\}$.

You see that solving a quadratic equation depends on our being able to factor the quadratic polynomial. Furthermore, our being able to factor a quadratic polynomial depends upon the standard form of the polynomial. If the standard form is the difference of two squares, then we can factor it.

Example 2. Solve the equation

$$x^2 - 2x + 2 = 0 .$$

Writing it in standard form, we have

$$x^2 - 2x + 2 = (x - 1)^2 + 1 .$$

This is not the difference of two squares, and we cannot factor "$x^2 - 2x + 2$" as a polynomial over the real numbers. The equation

$$(x - 1)^2 + 1 = 0$$

cannot have real number solutions, because $(x - 1)^2$ is always greater than or equal to 0 for every real number x. Why? Hence, $(x - 1)^2 + 1$ is greater than 0 for every real number x.

Problem Set 16-3

1. Factor the following polynomials over the real numbers, if possible.

(a)	$6x^2 - x - 15$	(e)	$x^2 - 3$
(b)	$x^2 + 4$	(f)	$9x^2 - 12x + 4$
(c)	$x^2 + 8x + 3$	(g)	$2(x - 1)^2 - 5$
(d)	$3(x - 2)^2 + 1$	(h)	$3x - 2x^2$

2. Solve the following quadratic equations.

(a)	$x^2 + 6x + 4 = 0$	(d)	$x^2 = 2x + 4$
(b)	$2x^2 - 5x = 12$	(e)	$2x^2 = 4x - 11$
(c)	$x^2 + 4x + 6 = 0$	(f)	$12x^2 - 8x = 15$

3. Find the coordinates of the vertex of the parabola whose equation is

$$y = -3x^2 + 6x - 5 .$$

What is the largest value the polynomial "$-3x^2 + 6x - 5$" can have?

4. The polynomial "$x^2 - 8x + 21$" may never have a value less than what positive integer? May it have values greater than this integer? Are all the values of the polynomial integers?

5. Consider the polynomial

$$2x^2 - 4x - 1$$

and its standard form

$$2(x - 1)^2 - 3 .$$

Since $2 = (\sqrt{2})^2$ and $3 = (\sqrt{3})^2$, "$2(x - 1)^2 - 3$" is the difference of two squares and can be factored as a polynomial over the real numbers.

$$2(x - 1)^2 - 3 = \left(\sqrt{2}(x - 1) + \sqrt{3}\right)\left(\sqrt{2}(x - 1) - \sqrt{3}\right)$$

What is the truth set of the equation

$$2x^2 - 4x - 1 = 0 ?$$

6. The perimeter of a rectangle is 94 feet, and its area is 496 square feet. What are its dimensions?

7. An open box is constructed from a rectangular sheet of metal 8 inches longer than it is wide as follows: out of each corner a square of side 2 inches is cut, and the sides are folded up. The volume of the resulting box is 256 cubic inches. What were the dimensions of the original sheet of metal?

8. Draw graphs of the following open sentences.
   (a) $y < x^2 + 6x + 5$
   (b) $y = 4$ and $y = 3x^2 - 12x + 13$
   (c) $y > 3x - 2x^2$
   (d) $y = x^2 - 6|x| + 5$.

9. A leg of a right triangle is 1 foot longer than the other leg and 8 feet shorter than the hypotenuse. Find the length of the sides of the right triangle.

10. A rope hangs from the window of a building. If pulled taut vertically to the base of the building there is 8 feet of rope lying slack on the ground. If pulled out taut until the end of the rope just reaches the ground, it reaches the ground at a point which is 28 feet from the building. How high above the ground is the window?

11. The hypotenuse of a right triangle is 3 units and the legs are equal in length. Find the length of a leg of the triangle.

12. Find the length of a diagonal of a square if the diagonal is 2 inches longer than a side.

13. The length of a rectangular piece of sheet metal is 3 feet more than the width. If the area is $46\frac{3}{4}$ square feet, find the length.

14. The sum of two numbers is 9 and the difference of their squares is 25 . Find the numbers.

15. The sum of 14 times a number and the square of the number is 11 . Find the number.

16. John drove 336 miles to Chicago, bought a new car and returned the next day by the same route. On the return trip his average speed was 6 miles per hour slower and took 1 hour longer than the original trip. Find his average speed each way.

17. The sum of a number and its reciprocal is 4 . Find the number.

Review Problems

1. Draw the graphs of the following polynomials:
   (a) $3x^2$
   (b) $3x^2 + 3$
   (c) $3(x - 3)^2$
   (d) $3x(x - 3)$
   (e) Explain how the graph of (d) can be obtained from the graph of (a) .

2. Given the graph of $y = x^2$.
   (a) Write an equation of the graph obtained by rotating the graph of $y = x^2$ one-half a revolution about the x-axis.
   (b) Write the equation of the graph obtained by moving the graph of $y = x^2$ vertically upward 3 units.
   (c) Write the equation of the graph obtained by moving the graph of $y = x^2$ horizontally 2 units to the left.
   (d) Write the equation of the graph obtained by moving the graph of $y = x^2$ one unit to the right and two units down.

3. Name the graphs of the following if possible; that is, "parabola", "line", "pair of lines", "point", etc.

(a) $y = x$

(b) $y = x^2$

(c) $y^2 = x^2$

(d) $y = 0 \cdot x^2$

(e) $x - y = 4$

(f) $x^2 - y = 4$

(g) $y = (x - 3)^2$

(h) $0 = (x - 3)^2$

(i) $x - 3 = y$ and $2x - y = 3$

4. For each of the following find:

(i) the points where the graph crosses the y-axis,

(ii) the points where the graph crosses the x-axis,

(iii) the largest (or smallest) value of the expression, if there is one.

(a) $x^2 + 2x - 8$

(b) $x^2 + 2x + 3$

(c) $-x^2 + 4$

(d) $-x + 4$

(e) $12 + x - x^2$

(f) $|x|$

5. Solve

(a) $x^2 + 21 = 10x$

(b) $x^2 = 2x + 1$

(c) $x + 6x^2 = 1$

(d) $35x^2 - 51x + 18 = 0$

(e) $x^2 + 1 = 4x$

(f) $x^2 + 2x + 2 > 0$

6. The sum of two numbers is 9. Find the numbers and their product if the product is the largest possible.

7. A boat manufacturer finds that his cost per boat in dollars is related to the number of boats manufactured each day by the formula,

$$c = n^2 - 10n + 175 .$$

Find the number of boats he should manufacture each day so that his cost per boat is smallest.

Chapter 17

FUNCTIONS

17-1. The Function Concept

Are you good at explaining things to other people? How would you explain to your younger brother exactly how to find the cost of sending a first-class parcel through the mails?

Let us say that you first go to your postmaster and learn these facts about first-class mail: A parcel weighing one ounce or less requires 4 cents postage; if it weighs more than one ounce and less than or equal to two ounces, it requires 8 cents postage, etc. The Post Office will not accept a parcel weighing more than 20 pounds for first-class mailing.

You would probably first explain to your brother that he should weigh his parcel carefully and find the number representing the weight in ounces. To what set of numbers will this number belong? Describe this set exactly. Now you will explain how to determine the amount of postage required. This will be a number in cents. To what set of numbers will this number belong? Describe this set exactly. If your brother's parcel weighs $3\frac{3}{4}$ ounces, what will the postage cost in cents? How much if it weighs 20 pounds and 15 ounces? (Remember the restriction on the weight of the parcel.)

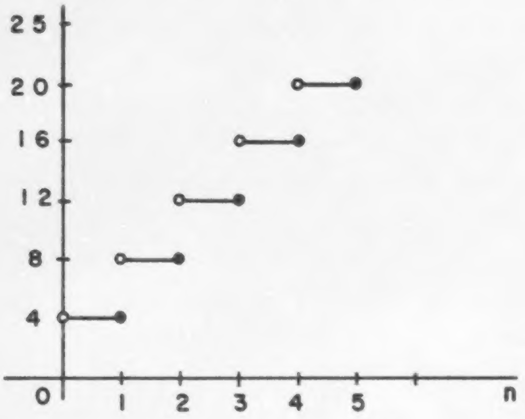
The problem of finding the amount of first-class postage really is a problem of pairing off the numbers of two sets. The numbers of the first set are the real numbers between 0 and 320 representing weights of parcels in ounces. The numbers of the second set are positive integers between 0 and 1280 representing costs of postage in cents. What you are really explaining to your brother is the description of these two sets and the rule which tells him how to take a given number of the first set and associate with it a number of the second set.

Your brother may ask you for a "formula" (to you this would mean an "expression in one variable") which would automatically give him the amount of postage for each number n of ounces. Can you find such a formula which assigns to each real number n between 0 and 320 the number of cents in the required postage? Would

$$4n$$

be such a formula? What is wrong with it?

If you can't find a formula, possibly you can satisfy your brother with a graph which will tell him at a glance what the postage costs. Let us draw a portion of such a graph (for n in ounces from $n = 0$ to $n = 5$):



Interpret the meanings of the circled points and the heavy dots on the graph. How would you explain to your brother how to use this graph to find the number of cents associated with $3\frac{1}{4}$ ounces? With 4 ounces?

Maybe he would understand the postage problem better if you drew up a table for him. Let us say that his scales read to the nearest $\frac{1}{4}$ ounce. You fill in the missing numbers in the table.

First Class Postage

ounces	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$	...
cents														...

You need not feel disappointed in not being able to find a formula for this association. There are many associations of numbers which cannot be described with an expression in one variable. The important point is whether the association can be described in any way, whether it be in terms of a verbal description a graph, a table, or an expression in one variable.

There are many places in the preceding chapters where we have had occasion in one way or another to associate a real number with each element of a given set. When an idea such as this turns up in such a variety of situations, it becomes worthwile to separate the idea out and study it carefully for its own sake. It is for this reason that we now make a special study of associations of real numbers of the kind illustrated in the above postage problem. First let us examine some more situations which involve such associations.

Problem Set 17-1a

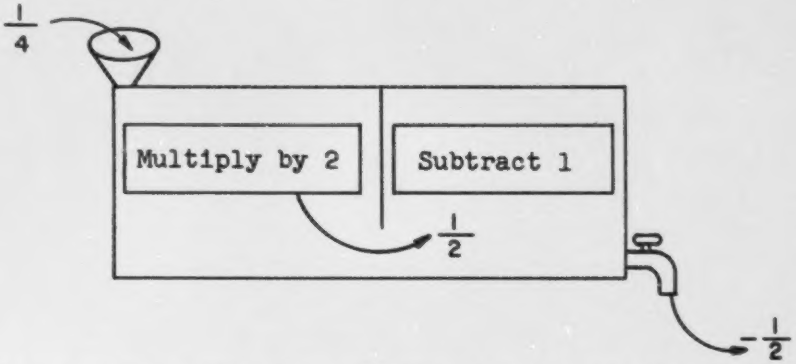
1. In each of the following, describe carefully the two sets and the rule which associates with each element of the first set an element of the second set.

(a)

Positive integer n	1	2	3	4	5	6	7	8	9	10	...
nth odd integer	1	3	5	7	9						

(Fill in the missing numbers. What number is associated with 13 ? With 1000 ?)

(b) Imagine a special computing machine which accepts any positive real number, multiplies it by 2 , subtracts 1 from this, and gives out the result.



(If you feed this machine the number 17, what will come out? What number does the machine associate with 0 ? With -1 ?)

(c) Draw two parallel real number lines and let the unit of measure on the upper line be twice that on the lower line. Then slide the lower line so that its point 1 is directly below the point 1 on the upper line. Now for each point on the upper (first) line there is a point directly below on the second line. (What number is below -13 ? Below 13 ? What number is associated with 1000 by this arrangement?)

(d) Draw a line with respect to a set of coordinate axes such that its slope is 2 and its y-intercept number is -1 . For each number a on the x-axis there is a number b on the y-axis such that (a, b) are the coordinates of a point on the given line. (If we pick -1 on the x-axis, the line associates with -1 what number on the y-axis? What number is associated with $-\frac{1}{2}$? With 13 ?)

(e) For each real number t such that $|t| < 1$, use the linear expression "$2t - 1$" to obtain an associated number. (What number does this expression associate with $-\frac{2}{3}$? With 2?)

(f) Given any negative real number, multiply it by 2 and then subtract 1. (What number does this verbal instruction associate with -13? With 0?)

2. In each of the following, describe the two sets involved and state verbally the rule which associates the elements of the sets. Tell, in each case, how many elements the rule associates with each element of the first set.

   (a) To each real number c such that $c < 1$, assign a number
   $$2c - 5 .$$

   (b) To each real number d, assign a number e such that (d, e) is a solution of the sentence "$d = |e|$".

   (c) To each real number x, assign a number y such that (x, y) is a solution of the equation
   $$y = 3x + 7 .$$

   (d) To each integer p, assign a number q such that (p, q) is a solution of the sentence "$p > q$".

   (e) To each rational number u, assign a number v such that (u, v) is a solution of the equation
   $$v^2 = u .$$

3. Give a precise verbal description of the association of weight and cost of a first-class parcel.

4. Describe the workings of a machine that weighs a parcel and automatically places the proper first-class postage on the parcel.

Some associations, such as in problems 2(b), (d), and (e), assign to each number in the first set more than one number in the second set. You will notice, however, that in all our other problems and examples, each number selected from the first set was associated with exactly one number of the second set.

This is the important idea we want to study. We call such an association a function.

> Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a function. The given set is called the domain of definition of the function, and the set of assigned numbers is called the range of the function.

It is very important to understand that two different rules give the same function if, and only if, they involve the same domain of definition and determine the same association of numbers. Thus, the functions of problems 1(d) and (e) are the same, even though they are described differently. But the functions of problems 1(a) and (b) are different because they have different domains of definition.

Now we see that a function may be described in many ways: By a table, as in problem 1(a) ; by a machine, as in problem 1(b) ; by a diagram, as in problem 1(c) ; by a graph, as in problem 1(d) ; by an expression in one variable, as in problem 1(e) ; or by a verbal description, as in problem 1(f) .

For our purpose, the most important way of describing a function is by an expression in one variable, since it allows us to use algebraic methods to study the function. On the other hand, it should be realized that a function need not, and in many cases cannot, be described by an expression in one variable. (Recall the example of the first class postage.) The graphical method is also important because it enables us to visualize certain properties of functions.

Problem Set 17-1b

1. Which of the statements in Problem Set 17-1a describe functions? If any do not, explain why not.

2. For those statements in Problem 2 of Problem Set 17-1a which describe functions, write if possible the rule in the form of an expression in x, where x belongs to the domain of definition. For example, in problem 2(a) the rule is given by $2x - 5$, where x is a real number less than 1.

3. In each of the following, describe (if possible) the function in two ways: (i) by a table, (ii) by an expression in x. In each case, describe the domain of definition.

   (a) With each day associate the income of the ice cream vendor in Chapter 6.

   (b) With each positive integer associate its remainder after division by 5.

   (c) To each positive real number assign the product of $\frac{1}{3}$ and two more than the number.

   (d) With each positive integer n associate the n^{th} prime.

   (e) Associate with each day of the year the number of days remaining in the (non-leap) year.

   (f) Associate with the number of dollars invested at 6% for one year the number of dollars earned as interest.

   (g) Associate with each length of the diameter of a circle the length of the circumference.

   (h) Draw two identical parallel number lines and slide the lower line so that its 0 point is directly under the point 1 of the upper line. Then rotate the lower line one-half revolution about its 0 point. Now associate with each number on the upper line the number directly below it on the rotated lower line.

4. With each positive integer greater than 1 associate the smallest factor of the integer (greater than 1). Form a table of ten of the associated pairs given by this function. What integers are associated with themselves?

5. The cost of mailing a package is determined by the weight of the package to the greatest pound. This can be described as: To every positive real number (weight in pounds) assign the integer which is closest to it and greater than or equal to it. Does this describe a function? Can it be represented by an expression in one variable? What is the domain of definition? (Note that the Post Office will not accept a package which weighs more than 32 pounds.) What number does this rule assign to 3.7 ? To 5 ?

6. Assign to each real number x the number -1 if x is rational and the number 1 if x is irrational. What numbers are assigned by this rule to $-\pi$, $-\frac{3}{2}$, $-\frac{\sqrt{2}}{2}$, 0, $\frac{1}{2}$, $\sqrt{2}$, $\frac{\pi}{2}$, 10^6 ? Can you represent this function any way other than by the verbal description?

7. Sometimes the domain of definition of a function is not stated explicitly but is understood to be the largest set of real numbers to which the rule for the function can be sensibly applied. For example, if a function is described by the expression $\frac{1}{x + 2}$, then, unless stated otherwise, its domain of definition is the set of all real numbers different from -2. (Why?) Similarly the domain of definition of the function defined by $\sqrt{x + 2}$ is the set of all real numbers greater than or equal to -2. (Why?) Find the domains of definition of the functions defined by the following expressions:

(a) $\frac{x}{x - 3}$ (c) $3 - \frac{1}{x}$ (e) $\sqrt{x^2 - 1}$

(b) $\sqrt{2x - 2}$ (d) $\sqrt{x^2}$ (f) $\frac{3}{x^2 - 4}$

8. In certain applications, the domain of definition of a function may be automatically restricted to those numbers which lead to meaningful results in the problem. For example, the area A of a rectangle with fixed perimeter 10 is given by $A = s(5 - s)$, where s is the length of a side in feet. The expression $s(5 - s)$ defines a function for all real s, but in this problem we must restrict s to numbers between 0 and 5. (Why?) What are the domains of definition of the functions involved in the following problems?

   (a) What amount of interest is earned by investing x dollars for a year at 4% ?

   (b) A triangle has area 12 square inches, and its base measures x inches. What is the length of its altitude?

   (c) An open top rectangular box is to be made by cutting a square of side x inches from each corner of a rectangular piece of tin measuring 10" by 8" and then folding up the sides. What is the volume of the box?

17-2. The Function Notation

We have been using letters as names of numbers and occasionally as names for certain expressions. In a similar way we shall use letters as names for functions. If f is a given function, and if x is a number in its domain of definition, then we shall designate the number which f assigns to x as $f(x)$. The symbol "$f(x)$" is read "f of x" (it is not f times x), and the number $f(x)$ is called the value of f at x .

The function notation is a very efficient one. Thus, when we wish to describe the function f: "To each real number x assign the real number $2x - 1$", we may write

$$f(x) = 2x - 1, \text{ for each real number } x .$$

Then,

$$f(\tfrac{1}{4}) = 2(\tfrac{1}{4}) - 1 = -\tfrac{1}{2} .$$

That is, f assigns to $\frac{1}{4}$ the number $-\frac{1}{2}$.

Similarly,

$$f(0) = 2(0) - 1 = -1 .$$

Also, $f(a) = 2a - 1$ for any real number a .

What real numbers are represented by

$$f(-\tfrac{4}{3}),\quad f(-\tfrac{1}{2}),\quad f(\tfrac{4}{3}),\quad f(s)$$

where s is a real number? If t is a real number, then

$$f(2t) = 2(2t) - 1 = 4t - 1 .$$

What real numbers are represented by:

$$f(-t),\quad -f(t),\quad 2f(t),\quad f(t-1),\quad f(t) - 1\ ?$$

Sometimes a function is defined in two or more parts, such as the function h defined by

$$h(x) = x, \text{ for each number } x \text{ such that } x \geq 0 ,$$

$$h(x) = -x, \text{ for each number } x \text{ such that } x < 0 .$$

This is a *single rule* and it defines *one* function, even though it involves two equations. It is customary to abbreviate this rule to the form

$$h(x) = \begin{cases} x, & x \geq 0 , \\ -x, & x < 0 . \end{cases}$$

What is the domain of definition of h ? The range of h ? Notice that $h(-3) = 3$ and $h(3) = 3$. In fact, we have worked with this function h before in the form

$$h(x) = |x|, \text{ for every real number } x .$$

Let us consider another function g defined by the rule:

$$\begin{cases} g(x) = -1, & \text{for each real number } x \text{ such that } x < 0 , \\ g(x) = 0, & \text{for } x = 0 , \\ g(x) = 1, & \text{for each real number } x \text{ such that } x > 0 . \end{cases}$$

It is important to understand that this is also a single rule for a single function which happens to be described in three parts. For convenience in writing, let us again abbreviate the above function g to the form:

$$g(x) = \begin{cases} -1, & x < 0 , \\ 0, & x = 0 , \\ 1, & x > 0 . \end{cases}$$

Notice that g assigns a number to every real number; hence, the domain of definition of g is the set of all real numbers. What is the range of the function? We see that $g(-5) = -1$ and $g(\pi) = 1$. What real numbers are represented by $g(-3.2)$, $g(0)$, $g(\frac{1}{2})$, $g(\sqrt{2})$? If $a > 0$, what is $g(a)$? What is $g(-a)$? If a is any non-zero real number, what is $g(|a|)$? Is it possible to write g in terms of a rule with a single equation, as we did for the function h in the preceding example?

Problem Set 17-2

1. Given the function F defined as follows:

$$F(x) = 2 - \frac{x}{2} \text{ for each real number } x .$$

What real numbers are represented by:

(a) $F(-2)$
(b) $-F(2)$
(c) $F(-\frac{1}{2})$
(d) $F(1) - 1$
(e) $F(0)$
(f) $|F(-6)|$
(g) $F(|-6|)$
(h) $F(t)$, for any real number t
(i) $F(\frac{t}{2})$
(j) $F(2t)$
(k) $F(\frac{1}{t})$

2. Given the function G defined by:

$$G(t) = |t| \text{ for each real number } t .$$

What is the range of G ? What real numbers are represented by:

(a) $G(0)$

(b) $G(a) - G(-a)$, for any real number a .

(c) $\frac{G(-3)}{3}$

3. Consider the function h defined by:

$$h(t) = \begin{cases} -1, & t < 0 , \\ 0, & t = 0 , \\ 1, & t > 0 . \end{cases}$$

How does this function h differ from the function g defined in Section 17-2? (Notice that the same function may have different names and involve different variables, provided the rule and the domain remain the same.)

4. Consider the function k defined by:

$$k(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 , \\ 0, & x = 0 . \end{cases}$$

Show that k is the same function as the function g defined in Section 17-2.

5. Given the function H defined by:

$$H(z) = z^2 - 1, -3 < z < 3 .$$

What real numbers are represented by:

(a) $H(2)$

(b) $H(\frac{1}{3})$

(c) $H(-\frac{1}{3})$

(d) $-H(-2)$

(e) $H(-1) + 1$

(f) $H(3)$

(g) $H(a)$, for any real number a such that $-3 < a < 3$.

(h) $H(t - 1)$, for any real number t such that $-2 < t < 4$.

(i) $H(t) - 1$, for any real number t such that $-3 < t < 3$.

6. Consider the function Q defined by:

$$Q(x) = \begin{cases} -1, & -1 \leq x < 0, \\ x, & 0 < x \leq 2. \end{cases}$$

(a) What is the domain of definition of Q ?

(b) What is the range of Q ?

(c) What numbers are represented by

$$Q(-1),\quad Q(-\tfrac{1}{2}),\quad Q(0),\quad Q(\tfrac{1}{2}),\quad Q(\tfrac{3}{2}),\quad Q(\pi) \quad ?$$

(d) If R is defined by

$$R(z) = \begin{cases} z, & 0 < z \leq 2, \\ -1, & -1 \leq z < 0, \end{cases}$$

is R a different function from Q ?

7. Let F be the function defined in Problem 1. What is the truth set of each of the following sentences?

(a) $F(x) = -1$ (c) $F(x) = -\frac{1}{2}$ (e) $F(x) > 2$

(b) $F(x) < 0$ (d) $F(x) = x$ (f) $F(x) \leq 1$

8. Let G be the function defined in Problem 2. Draw the graphs of the truth sets of the following sentences.

(a) $G(x) = 1$ (c) $G(x) \leq 1$

(b) $G(x - 1) = 1$ (d) $G(x + 1) > 2$

9. Describe how each of the following pairs of functions differ, if at all:

(a) $f(x) = x - 2$; $F(x) = \frac{x^2 - 4}{x + 2}$

(b) $g(x) = x^2 - 1$; $G(t) = \frac{t^4 - 1}{t^2 + 1}$

17-3. Graphs of Functions

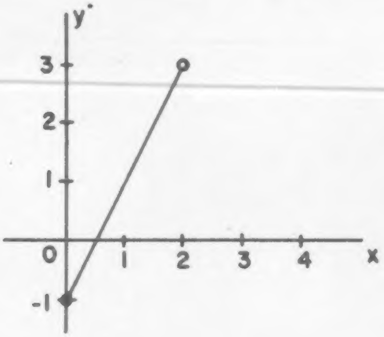
One way to represent a function is by means of a graph, as we have seen early in this chapter. When a function f is defined, the graph of f is the graph of the truth set of the equation

$$y = f(x) .$$

Example 1. Draw the graph of the function f defined by:

$$f(x) = 2x - 1, \ 0 \leq x < 2 .$$

This is the graph of the equation $y = 2x - 1, \ 0 \leq x < 2$.



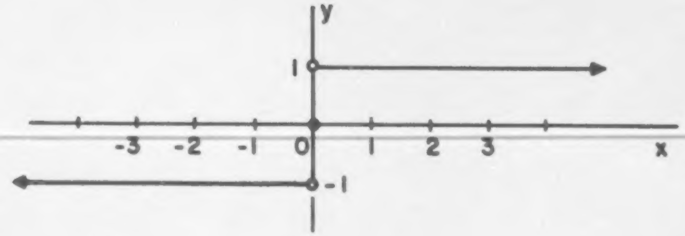
Is this the same as the graph of the function F defined by

$$F(x) = 2x - 1, \quad -2 < x < 2 ?$$

Example 2. Draw the graph of the function g defined by:

$$g(x) = \begin{cases} -1, & x < 0 , \\ 0, & x = 0 , \\ 1, & x > 0 . \end{cases}$$

The graph of g is



Problem Set 17-3a

1. Draw the graphs of the functions defined as follows.

   (a) $T(s) = \frac{1}{3}s + 1, \quad -1 \leq s \leq 2$

   (b) $G(x) = |x|, \quad -3 \leq x \leq 3$

   (c) $U(x) = \begin{cases} -x, & -3 \leq x < 0 \\ x, & 0 \leq x < 3 \end{cases}$

   (d) $V(t) = t^2 - 1, \ -2 < t \leq 1$

   (e) $h(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

   (f) $H(z) = \frac{|z|}{z}$

2. What are the domains of definition and the ranges of the functions defined in Problem 1?

3. Draw the graph of the function q defined by

$$q(x) = \begin{cases} -1, & -5 \leq x < -1 , \\ x, & -1 \leq x < 1 , \\ x^2, & 1 < x \leq 2 , \end{cases}$$

4. Give a rule for the definition of the function whose graph is the line extending from (-2, 2) to (4, -1), including end points.

5. Give a rule for the definition of the function whose graph consists of two line segments, one extending from (-1, 1) to (0, 0) with end points included, and the other extending from (0, 0) to (2, 1) with end points excluded. What are the domain of definition and range of this function?

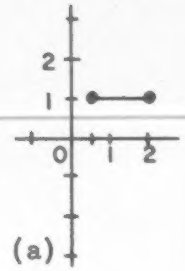
6. Draw the graph of a function f which satisfies all of the following conditions over the domain of definition, $-2 \leq x \leq 2$:

$$\begin{aligned} f(-1) &= 2 , \\ f(0) &= 0 , \\ f(1) &= 0 , \\ f(2) &= 2 , \\ f(x) &< 0 \quad \text{for} \quad 0 < x < 1 . \end{aligned}$$

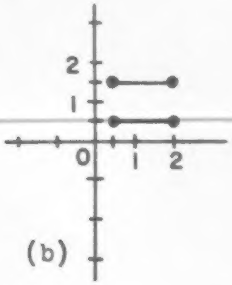
Now that we know how to draw the graph of a function, it is natural to ask whether a given set of points in the plane is the graph of some function. Draw several sets of points and then ask yourself what the definition of a function requires of its graph: It requires that to each abscissa in the domain of definition there be exactly one ordinate assigned by the function. Thus, for each number a in the domain of definition of the function, how many points on its graph have this number as abscissa? If a vertical line is drawn through the graph of a function, in how many points will the line intersect the graph? How would you state the rule for a function in terms of its graph?

Problem Set 17-3b

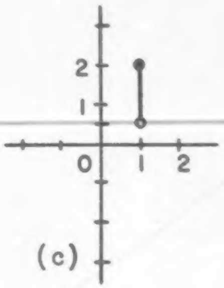
1. Consider the sets of points, coordinate axes not included, indicated in the following figures.



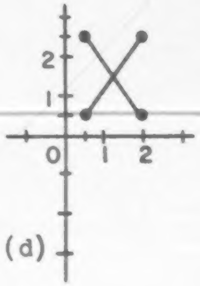
(a)



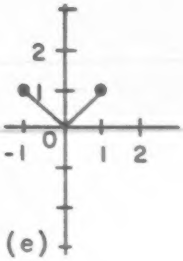
(b)



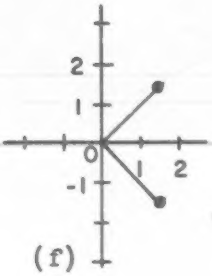
(c)



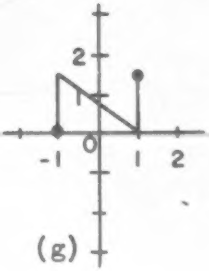
(d)



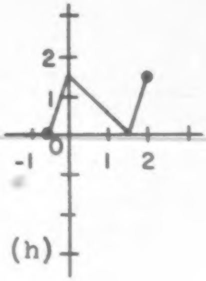
(e)



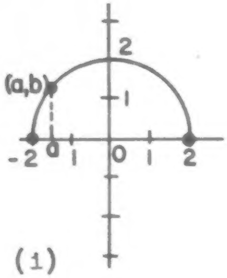
(f)

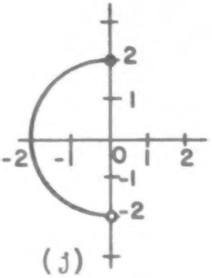


(g)

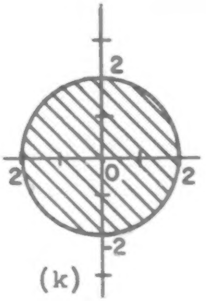


(h)

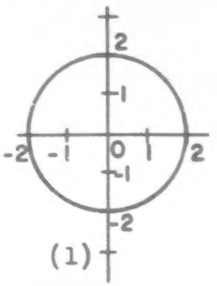

2
(a,b)
1
-2
a
1
0
1
2
(i)



(j)



(k)

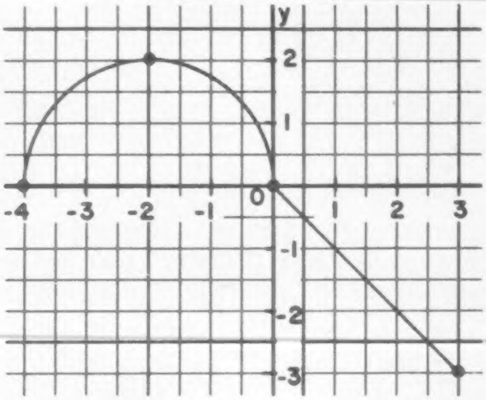


(l)

Which of the above figures is the graph of some function?
[sec. 17-3]

Give the reason for your answer in each case. As an illustration, consider figure (i). This is the graph of a function f whose domain of definition is the set of all x such that $-2 \leq x \leq 2$. The rule for the function can be stated as follows: If $-2 \leq a \leq 2$, then $f(a) = b$, where (a, b) is the (unique !) point on the graph with abscissa equal to a .

2. The accompanying figure is the graph of a function h . From the graph estimate
   (a) $h(-3)$, $h(0)$, $h(2)$;
   (b) the domain of definition of h;
   (c) the range of h.



3. Let G denote a set of points in the plane which is the graph of some function g .
   (a) For each x in the domain of definition of g , explain how to use the graph to obtain $g(x)$.
   (b) How do you obtain the domain of definition of g from the graph of G ?
   (c) Show that if (a, b) and (c, d) are any two distinct points of the graph G, then $a \neq c$.

4. Let G be any set of points in the plane with the property that, if (a, b) and (c, d) are any two distinct points of G, then $a \neq c$. Show that G is the graph of a function.

5. Draw the graph of the equation $y^2 = x$, for $0 \leq x < 4$. Is this the graph of some function?

17-4. Linear Functions

A function whose graph is a straight line (or a portion thereof) is called a linear function. You have already worked with linear functions in Chapter 14, but there they were studied in the form of linear expressions. Can each straight line in the plane be considered as the graph of some linear function? How about the line whose equation is $x = 2$? Can each linear function be represented by an expression in one variable? What is the general form of such an expression? (Recall the y-form of the equation of a line.)

Problem Set 17-4

1. If f is a linear function, then there are real numbers A and B such that $f(x) = Ax + B$ for every x in the domain of definition of f.
   - (a) Describe the graph of f if $A = 0$.
   - (b) Describe the graph of f if $A = 0$ and $B = 0$.
   - (c) Determine A and B if the graph of f is the line segment joining (-3, 0) and (1, 2), including end points.
   - (d) What is the domain of definition of the function in part (c)?
   - (e) Determine A and B if the graph of f is the line segment joining (-1, 1) and (3, 3) excluding end points.

(f) What is the slope and y-intercept of the graph of the function in part (e)?

(g) What is the domain of definition of the function in part (e)?

2. If L is the complete line containing the two points $(-3, 1)$ and $(1, -1)$, describe the function h whose graph consists of the points (x, y) of L such that

$$-2 < y < 2 .$$

3. Which of the following expressions describe a linear function (i.e., with a straight-line graph)?

(a) $-(x - 2)$

(b) $|x - 2|$

(c) $\frac{1}{x - 2}$

(d) $|x| - 2$

(e) $(-x) - 2$

(f) $x^2 - 2$

4. Let f be the linear function defined by:

$$f(x) = x - 2, \text{ for every real number } x .$$

Write each expression in Problem 3 as a function g in terms of the given function f. Example: The expression (a) describes a function g such that

$$g(x) = -f(x), \text{ for every real number } x .$$

5. How are the graphs of f and g related in Problem 4, parts (a) and (e) ? Draw each pair with reference to a separate set of coordinate axes.

6. If F and G are linear functions defined for every real x by

$$F(x) = -3x + 2, \quad G(x) = 2x - 3 ,$$

explain how the graph of the sentence

$$\Big(y - F(x)\Big)\Big(y - G(x)\Big) = 0$$

is related to the graphs of F and G. (Do this without drawing the graphs of F and G .)

17-5. Quadratic Functions

We described a linear function in terms of a linear expression in one variable; i.e., any linear function f can be defined by

$$f(x) = Ax + B ,$$

where A, B are real numbers. It is natural to define a quadratic function as one which is expressed in terms of a quadratic polynomial in one variable,

$$Ax^2 + Bx + C ,$$

where A, B, C are real numbers. If $A = 0$, the quadratic polynomial is reduced to the linear case; hence, we shall assume throughout the remainder of this chapter that $A \neq 0$.

Example 1. Define the function g by:

$$g(x) = 2x^2 - 3x + 1, \text{ for every real number } x .$$

Then

$$g(0) = 1, \quad g(\tfrac{3}{2}) = 2(\tfrac{3}{2})^2 - 3(\tfrac{3}{2}) + 1 = 1 ,$$

$$g(t) = 2t^2 - 3t + 1, \quad g(2t) = 2(2t)^2 - 3(2t) + 1$$

$$= 8t^2 - 6t + 1 ,$$

$$g(t - 1) = 2(t - 1)^2 - 3(t - 1) + 1 = 2t^2 - 7t + 6 .$$

Also, since $g(x) = (2x - 1)(x - 1)$, it follows that

$$g(\tfrac{1}{2}) = 0 \text{ and } g(1) = 0 . \quad \text{(Why?)}$$

Notice that

$$g(|-2|) = 2|-2|^2 - 3|-2| + 1 = 3 ,$$

whereas

$$|g(-2)| = |2(-2)^2 - 3(-2) + 1| = 15 .$$

Problem Set 17-5a

1. Let f be the quadratic function defined by

   $f(x) = x^2 - 3x - 21$, for every real number x,

   and g the quadratic function defined by

   $g(x) = 3x^2 - 2, -3 < x < 3$.

   (a) Determine $f(-2)$, $f(-\frac{1}{2})$, $f(0)$, $f(\frac{3}{4})$, $f(3)$; $f(a)$, $f(\frac{a}{2})$, $f(a + 1)$, where a is any real number.

   (b) Determine $g(-2)$, $g(-\frac{1}{2})$, $g(0)$, $g(3)$; $g(2t - 1)$, $-1 < t < 2$.

   (c) Find the truth set of the sentence "$f(x) = 0$".

   (d) Draw the graph of the sentence "$f(x) < 0$".

   (e) Determine $f(t) + g(t)$, $-3 < t < 3$.

   (f) Determine for the real number a $f(a) + 3$, $f(a + 3)$, $3f(a)$, $f(3a)$.

   (g) Are all the resulting polynomials in part (f) quadratic polynomials in a?

   (h) Determine $f(t)g(t)$, $-3 < t < 3$.

   (i) Is the resulting polynomial in (e) a quadratic polynomial in t? How about the resulting polynomial in (h)?

2. Describe the functions involved in the following problems. State the domains of definition, and solve the problems.

   (a) What is the area A of a triangle if the length of the base is b inches and the altitude is 10 inches longer than the base?

   (b) What is the product P of two positive numbers if the larger plus twice the smaller, s, is 120?

(c) 120 feet of wire is to be used to build a rectangular pen along the wall of a large barn, the wall of the barn forming one side of the pen. If L is the length of the side of the pen parallel to the wall of the barn, find the area A of the pen.

3. Draw the graph of the quadratic function f defined by:

(a) $f(x) = x^2 + x - 1, -3 \leq x \leq 2$

(b) $f(x) = 3x^2 - 3, -2 < x < 2$

(c) $f(x) = -x^2 + 1, -2 \leq x < 3$

In problems 4 through 8, refer to Figure 1. and the graph of $y = x^2$, and also refer to Section 16-1.

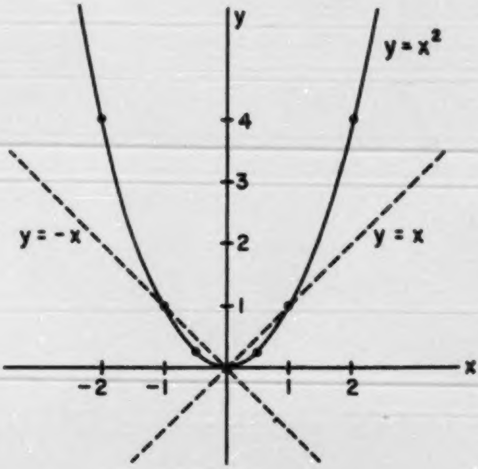


Figure 1.

4. For any real number x, $x^2 \geq 0$. (Why?) Also, $x^2 = 0$ if and only if $x = 0$. Explain why the graph of $y = x^2$ lies entirely above the x-axis and touches the x-axis at a single point (0, 0).

5. For any real number x,

$$(-x)^2 = x^2 .$$

If (a, b) is a point on the graph of $y = x^2$, prove that $(-a, b)$ is also on the graph. (This means that the portion of the graph in Quadrant II can be obtained by rotating the portion in Quadrant I about the y-axis. We say that "the graph of $y = x^2$" is symmetric about the y-axis.)

6. If x is any real number such that $0 < x < 1$, then $x^2 < x$. (Why?) Show that the portion of the graph of $y = x^2$, for $0 < x < 1$, lies below the graph of $y = x$.

7. If $1 < x$, then
$$x < x^2 . \quad \text{(Why?)}$$
Show that the portion of the graph of $y = x^2$, for $1 < x$, lies above the line $y = x$.

8. If a and b are real numbers such that $0 < a < b$, then
$$a^2 < b^2 . \quad \text{(Why?)}$$
Show that the graph of $y = x^2$ rises steadily as we move to the right from 0.

9. Show that a horizontal line will intersect the graph of $y = x^2$ in at most two points.

10. Choose any point (a, a^2) on the graph of $y = x^2$. What is the slope of the line containing $(0, 0)$ and (a, a^2)? As we choose points of the graph close to the origin (a close to 0) what happens to the slope of this line? Can you explain why the graph of $y = x^2$ is flat near the origin?

Problems 4 through 10 justify the graph of $y = x^2$ drawn in Figure 1. This graph is an example of a parabola. The point $(0, 0)$ is called its vertex, and the line $x = 0$ is called its axis.

With our knowledge of the graph of $y = x^2$, we can obtain graphs of other quadratic functions. This was done in Section 16-1 for particular quadratic functions. Let us verify these extensions of the graph of $y = x^2$ to the graphs of $y = Ax^2 + Bx + C$ for real numbers A, B, C, where $A \neq 0$.

Problem Set 17-5b

1. Describe how the graph of $y = ax^2$ differs from the graph of $y = x^2$ in each of the following cases.

   (a) $0 \leq a < 1$

   (b) $a > 1$

   (c) $-1 < a < 0$

   (d) $a < -1$

   (e) $|a|$ very large.
   (Refer to Figure 2.)

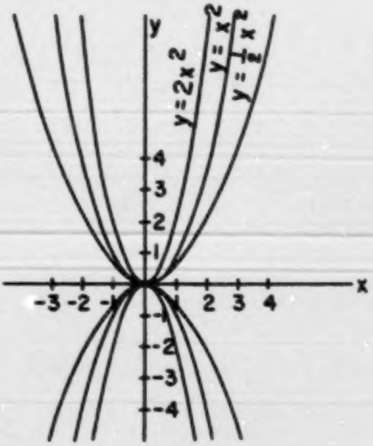


Figure 2.

2. Describe how the graph of $y = x^2 + k$ differs from the graph of $y = x^2$ in each of the following cases.

   (a) $k > 0$

   (b) $k < 0$
   (Refer to Figure 3.)

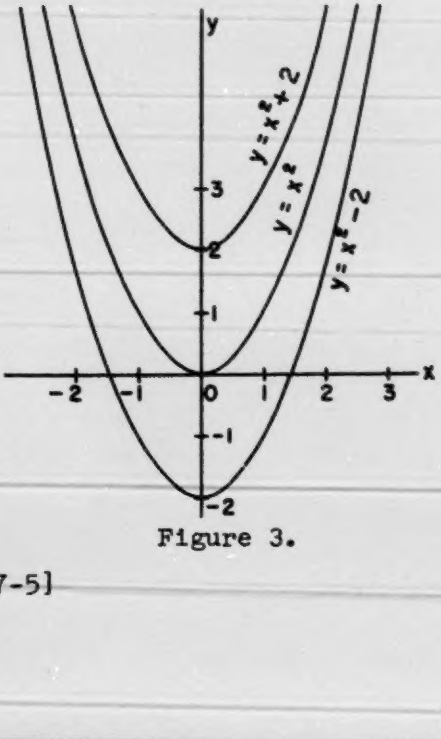


Figure 3.

3. Describe how the graph of $y = (x - h)^2$ differs from the graph of $y = x^2$ in the cases:

   (a) $h > 0$, (b) $h < 0$.
   (Refer to Figure 4.)

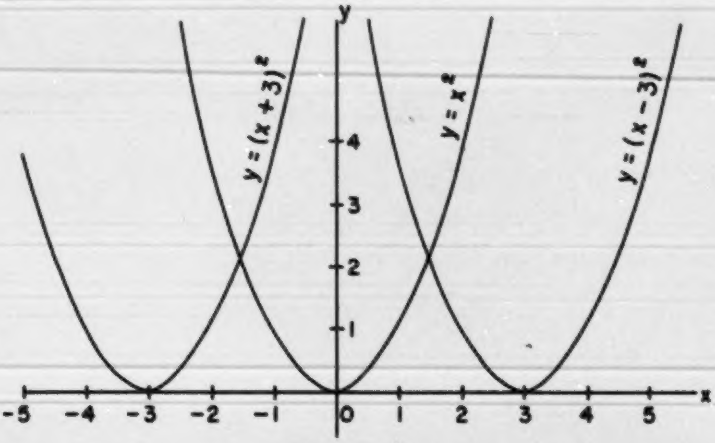


Figure 4.

4. Using the results of Problems 1 to 3, describe without drawing their graphs the appearances of the following parabolas.

   (a) $y = (x + 1)^2$
   (b) $y = -3x^2$
   (c) $y = x^2 - 3$
   (d) $y = -(x - 1)^2$
   (e) $y = 2(x - 2)^2$
   (f) $y = (x + 1)^2 + 1$
   (g) $y = 2(x - 1)^2 - 1$
   (h) $y = -2(x + 1)^2 - 1$

5. If a, h, k are real numbers, discuss how the graph of $y = a(x - h)^2 + k$ can be obtained from the graph of $y = ax^2$. What is the vertex of the parabola $y = a(x - h)^2 + k$? What is the equation of the axis of this parabola?

6. What is an equation of a parabola whose vertex is $(-1, 1)$ and whose axis is the line $y = 1$? How many parabolas fulfill these conditions?

17-6. The Graph of $y = Ax^2 + Bx + C$

In Section 16-2 we learned how to write a quadratic polynomial $Ax^2 + Bx + C$ in the form $a(x - h)^2 + k$. We called this latter form the standard form of the polynomial. In the preceding section we learned how to draw the graph of the equation $y = a(x - h)^2 + k$. Thus, we have a method for drawing quickly the graph of any quadratic function.

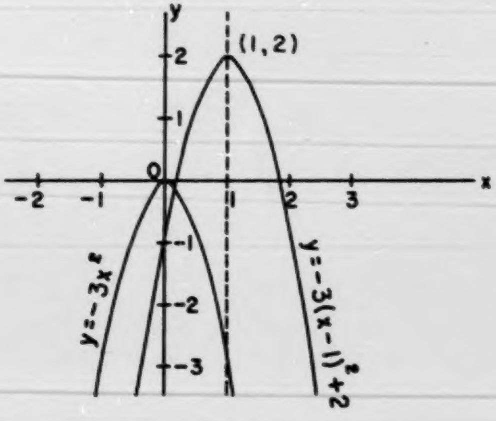
Example 1. Draw the graph of the function f defined by:

$$f(x) = -3x^2 + 6x - 1 .$$

By completing the square, we obtain:

$$-3x^2 + 6x - 1 = -3(x^2 - 2x + 1) - 1 + 3 = -3(x - 1)^2 + 2 .$$

The graph of $y = -3(x - 1)^2 + 2$ is obtained from the graph of $y = -3x^2$ as indicated:



To verify that the process of moving the graph of $y = -3x^2$ to the right 1 unit and upward 2 units will actually yield the graph of $y = -3(x - 1)^2 + 2$, let us argue as follows: Suppose that (a, b) are the coordinates of a point on the graph of the equation

$$y = -3(x - 1)^2 + 2 .$$

Then these coordinates must satisfy the equation; i.e.,

$$b = -3(a - 1)^2 + 2$$

is a true sentence. But then

$$b - 2 = -3(a - 1)^2$$

is also a true sentence. This final sentence asserts that the point with coordinates $(a - 1, b - 2)$ is on the graph of $y = -3x^2$. But what are the relative positions of the points $(a - 1, b - 2)$ and (a, b) ? From $(a - 1, b - 2)$ we must move 1 unit to the right and 2 units upward to arrive at (a, b) . That is precisely what we did to every point on the graph of $y = -3x^2$ to arrive at a point on the graph of $y = -3(x - 1)^2 + 2$.

Problem Set 17-6

1. Write the standard form and draw the graph of each of the following quadratic polynomials.

   (a) $x^2 - 6x + 10$

   (b) $4x^2 + 4x - 9$

   (c) $5x^2 - 3$

   (d) $-x^2 - x + \frac{9}{4}$

   (e) $4x^2 + 4cx + c^2$

   (f) $5x^2 - 3x - \frac{11}{20}$

2. For each quadratic polynomial of problem 1, find the points (if any) where the graph crosses the x-axis.

3. Prove the Theorem: Given any quadratic polynomial $Ax^2 + Bx + C$, there exist real numbers a, h, k such that

$$a(x - h)^2 + k = Ax^2 + Bx + C, \text{ for every real number } x .$$

The numbers a, h, k are related to the numbers A, B, C by the true sentences

$$a = A, \quad h = -\frac{B}{2A}, \; k = \frac{4AC - B^2}{4A}$$

*4. The problem of changing a quadratic polynomial, such as $-2x^2 - 4x + 1$, into standard form can also be handled as follows. Let us find numbers a, h, k (if possible) such that

$$a(x - h)^2 + k = -2x^2 - 4x + 1$$

for every real number x. By simplifying and regrouping the left member, we write

$$ax^2 - 2ah\,x + (ah^2 + k) = -2x^2 - 4x + 1 ,$$

for every real number x. Now we see at a glance that we must find a, h, k so that

$$a = -2, \quad -2ah = -4, \quad ah^2 + k = 1 . \quad \text{(Why?)}$$

If $a = -2$, then "$-2ah = -4$" is equivalent to "$4h = -4$", i.e., to "$h = -1$". Also, if $a = -2$ and $h = -1$, then "$ah^2 + k = 1$" is equivalent to "$-2 + k = 1$", i.e., to "$k = 3$". With $a = -2$, $h = -1$, and $k = 3$, we have

$$-2(x + 1)^2 + 3 = -2x^2 - 4x + 1 ,$$

for every real number x.

Using this method write each of the following in standard form.

(a) $3x^2 - 7x + 5$

(b) $5x^2 - 3x + \frac{13}{20}$

(c) $Ax^2 + Bx + C$, where A, B, C are real numbers.

17-7. Solutions of Quadratic Equations

Consider the three quadratic polynomials

$$x^2 + 2x - 3, \quad x^2 + 2x + 1, \quad x^2 + 2x + 3 ,$$

and their graphs shown in Figure 5.

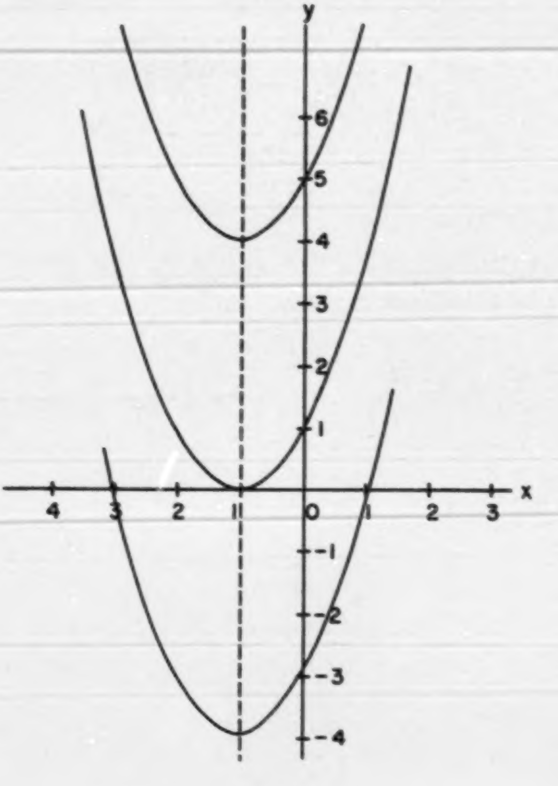


Figure 5.

Notice that the graph of $y = x^2 + 2x - 3$ crosses the x-axis in two points; the graph of $y = x^2 + 2x + 1$ touches the x-axis in a single point; and the graph of $y = x^2 + 2x + 3$ does not intersect the x-axis at all! What is the ordinate of any point on the x-axis? Another way of describing the intersections of these graphs with the x-axis is: The truth set of

$$x^2 + 2x - 3 = 0 \text{ is } \{-3, 1\},$$

$$\text{of } x^2 + 2x + 1 = 0 \text{ is } \{-1\},$$

$$\text{of } x^2 + 2x + 3 = 0 \text{ is } \emptyset.$$

In general, since the graph of

$$y = Ax^2 + Bx + C$$

is always a parabola (if $A \neq 0$), it seems evident that the truth set of the quadratic equation

$$Ax^2 + Bx + C = 0$$

will consist of two, one, or no real numbers according as the parabola intersects, touches, or does not intersect the x-axis.

We have already learned how to solve a quadratic equation whose left side can be factored as a polynomial over the integers. Now consider the quadratic polynomial

$$x^2 + 2x - 1 .$$

We know that it cannot be factored as a polynomial over the integers. (Why?) In Chapter 12 we learned how to write this as

$$\begin{aligned} x^2 + 2x - 1 &= (x + 1)^2 - 2 \\ &= (x + 1)^2 - (\sqrt{2})^2 , \end{aligned}$$

i.e., as the difference of two squares. Hence, we may factor $x^2 + 2x - 1$ as a polynomial over the real numbers:

$$x^2 + 2x - 1 = \left((x + 1) + \sqrt{2}\right)\left((x + 1) - \sqrt{2}\right).$$

(Verify this by multiplying the factors.) Then

$$x^2 + 2x - 1 = 0 ,$$

$$(x + 1 + \sqrt{2})(x + 1 - \sqrt{2}) = 0 ,$$

$$x + 1 + \sqrt{2} = 0 \text{ or } x + 1 - \sqrt{2} = 0 ,$$

$$x = -1 - \sqrt{2} \text{ or } x = -1 + \sqrt{2} ,$$

are all equivalent sentences, and so the truth set of

$$x^2 + 2x - 1 = 0 \text{ is } \left\{-1 - \sqrt{2},\ -1 + \sqrt{2}\right\}.$$

This example suggests a general procedure for determining whether a quadratic equation has real solutions, and, if so, for finding the solutions. We have shown that any quadratic equation

$$Ax^2 + Bx + C = 0$$

can be written in standard form

$$a(x - h)^2 + k = 0 .$$

Let us assume that a is positive. Otherwise, we may multiply both sides by (-1). The case in which k is a positive number can be disposed of quickly, because we have learned that the graph of $a(x - h)^2 + k$ lies entirely above the x-axis if a and k are positive. Then it cannot cross the x-axis. Hence, there are no real solutions of the equation if $k > 0$.

If $k = 0$, we saw that the graph of $a(x - h)^2$ touches the x-axis at the point $(h, 0)$. Hence, there is one real solution if $k = 0$.

This leaves the case in which k is a negative number. Now we may write

$$a(x - h)^2 + k = a(x - h)^2 - (-k) .$$

If $k < 0$, is there a real number whose square is $(-k)$? How do we factor the difference of two squares? Your result should be

$$a(x - h)^2 - (-k) = (\sqrt{a}(x - h) + \sqrt{-k})(\sqrt{a}(x - h) - \sqrt{-k}) .$$

Thus, if k is negative, the polynomial $a(x - h)^2 + k$ can always be factored as a polynomial over the real numbers, and the equation has two real solutions.

Example 1. Factor (a) $2x^2 + 3x - 1$, (b) $x^2 + 3x + 4$.

$$\text{(a)} \quad 2x^2 + 3x - 1 = 2(x + \tfrac{3}{4})^2 - \tfrac{17}{8} \qquad \text{(Verify this.)}$$

$$= 2\left((x + \tfrac{3}{4})^2 - \tfrac{17}{16}\right)$$

$$= 2\left((x + \tfrac{3}{4})^2 - \left(\tfrac{\sqrt{17}}{4}\right)^2\right)$$

$$= 2\left(x + \tfrac{3}{4} + \tfrac{\sqrt{17}}{4}\right)\left(x + \tfrac{3}{4} - \tfrac{\sqrt{17}}{4}\right) .$$

$$\text{(b)} \quad x^2 + 3x + 4 = (x + \tfrac{3}{2})^2 + \tfrac{7}{4} . \qquad \text{(Verify this.)}$$

Here k is a positive number, and we cannot factor this sum of two squares as a polynomial over the real numbers. In fact, $x^2 + 3x + 4$ can never assume the value 0 for any real number x, because $x^2 + 3x + 4$ is the sum of a non-negative number $(x + \frac{3}{2})^2$ and a positive number $\frac{7}{4}$.

Example 2. Solve the equation $x - 3x^2 + 7 = 0$.

The equations $x - 3x^2 + 7 = 0$,

$$3x^2 - x - 7 = 0 ,$$

$$3(x - \tfrac{1}{6})^2 - \tfrac{85}{12} = 0 , \quad \text{(Why?)}$$

$$(x - \tfrac{1}{6})^2 - \tfrac{85}{36} = 0 , \quad \text{(Why?)}$$

$$\left(x - \tfrac{1}{6} + \tfrac{\sqrt{85}}{6}\right)\left(x - \tfrac{1}{6} - \tfrac{\sqrt{85}}{6}\right) = 0 ,$$

$$x - \tfrac{1}{6} + \tfrac{\sqrt{85}}{6} = 0 \quad \text{or} \quad x - \tfrac{1}{6} - \tfrac{\sqrt{85}}{6} = 0 ,$$

$$x = \frac{1 - \sqrt{85}}{6} \quad \text{or} \quad x = \frac{1 + \sqrt{85}}{6} ,$$

are all equivalent. Hence, the truth set of

$$x - 3x^2 + 7 = 0 \text{ is } \left\{\frac{1 - \sqrt{85}}{6}, \frac{1 + \sqrt{85}}{6}\right\}.$$

Problem Set 17-7

1. Factor the following quadratic polynomials over the real numbers, if possible.

   (a) $t^2 - 10t + 26$

   (b) $6x^2 - x - 12$

   (c) $\frac{1}{2}x^2 + 4x + 6$

   (d) $4y^2 + 2y + \frac{1}{4}$

   (e) $x^2 + 7x + 14$

   (f) $2 - 2z - z^2$

   (g) $1 - 5x^2$

   (h) $7x^2 - \frac{14}{3}x + \frac{25}{9}$

   (i) $5v^2 - 5v - \frac{11}{4}$

   (j) $x^2 + (a + b)x + ab$, a and b any real numbers

2. Solve the following quadratic equations.

   (a) $4 - 3x^2 = 0$

   (b) $4 - x - 3x^2 = 0$

   (c) $4 - x + 3x^2 = 0$

   (d) $s^2 - s - \frac{1}{2} = 0$

   (e) $\frac{4}{5}t^2 + \frac{4}{5}t + \frac{1}{5} = 0$

   (f) $\frac{1}{3}y^2 + 2y - 3 = 0$

   (g) $-2y^2 + y - \frac{1}{2} = 0$

   (h) $3n^2 = 7n$

3. Consider the quadratic polynomial in standard form, $a(x - h)^2 + k$, where a, h, k are real numbers and $a \neq 0$.

   (a) State a rule for deciding whether or not this polynomial over the real numbers can be factored.

   (b) If a, h, k are integers, what conditions on these numbers guarantee that this polynomial over the integers can be factored?

   (c) State a rule for deciding whether the truth set of

   $$a(x - h)^2 + k = 0$$

   contains two, one, or no real numbers.

4. Translate the following into open sentences and solve.

   (a) The perimeter of a rectangle is 12 inches and its area is 7 square inches. What is the number x of inches in the length of its longer side?

   (b) One side of a right triangle is x inches and this side is 1 inch longer than the second side and 2 inches shorter than the hypotenuse. Find x .

   (c) The sum of two numbers is 5 and their product is 9 . What are the numbers?

5. Consider the general quadratic polynomial $Ax^2 + Bx + C$. Show that

   (a) $Ax^2 + Bx + C = A\left(\left(x + \frac{B}{2A}\right)^2 - \frac{B^2 - 4AC}{4A^2}\right)$

   (b) If $B^2 - 4AC < 0$, then $Ax^2 + Bx + C = 0$ has no real solution.

   (c) If $B^2 - 4AC = 0$, then $Ax^2 + Bx + C = 0$ has one real solution, $x = -\frac{B}{2A}$.

   (d) If $B^2 - 4AC > 0$, then $Ax^2 + Bx + C = 0$ has two real solutions,

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A} .$$

   (This latter sentence is called the quadratic formula for finding the solutions of the quadratic equation.)

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